The comonotonicity coefficient: a multivariate measure for positive dependence

Inge Koch & Ann De Schepper
University of Antwerp

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- Comonotonicity Coefficient $\kappa(\boldsymbol{X})$
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Introduction

Subject: to construct a measure which reflects the degree of comonotonicity in a random vector.

- for a long time independence assumption $\longrightarrow$ take dependence into account
  - Really model the dependence through e.g. *copulas*
  - NOT model the dependence but look at extremes to obtain upper and lower bounds e.g. *convex bounds by means of comonotonic vectors* $\longrightarrow$ useful when structure is "rather comonotonic"

We want to quantify the "rather comonotonic" .

Preliminaries (1)

- The cdf of a univariate random variable X: $F_X(x) = P(X \leq x)$.
- The (joint) cdf of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$: $F_{\mathbf{X}}(\mathbf{x}) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_n \leq x_n)$, with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$.
- This joint cdf captures all the information of the marginal cdf and the dependency structure.
- The set of all joint cdfs with the same marginal cdfs $F_1, F_2, \ldots, F_n$, is called the Fréchet space $\mathcal{R}_n(F_1, F_2, \ldots, F_n)$.

Preliminaries (2)

- Fréchet upper bound:
  $F^C_{\mathbf{X}}(\mathbf{x}) = Min\{F_1(x_1), F_2(x_2), \ldots, F_n(x_n)\}$
  $\Rightarrow F_{\mathbf{X}}(\mathbf{x}) \leq F^C_{\mathbf{X}}(\mathbf{x})$
- $U \sim Uniform[0,1]$, $F^C_{\mathbf{X}}$ is the cdf of:
  $(F_1^{-1}(U), F_2^{-1}(U), \ldots, F_n^{-1}(U))$
- where
  $F_i^{-1}(p) = inf\{x \in |F_i(x) \geq p\}, \quad p \in [0,1].$
- Random vectors wich have $F^C_{\mathbf{X}}(\mathbf{x})$ as cdf are comonotonic.

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Preliminaries (3)

Definition

A random vector $\mathbf{X}$ is comonotonic if and only if there exists a random variable Z and non-decreasing functions $t_1, t_2, \ldots, t_n$, such that

$$\mathbf{X} =_d (t_1(Z), t_2(Z), \ldots, t_n(Z)).$$

Notation: $(X_1^c, X_2^c, \ldots, X_n^c)$ for the comonotonic random vector with marginal distributions $F_1, F_2, \ldots, F_n$.

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Preliminaries (4)

Definition

A random vector **X** is independent if and only if its cdf is given by:

$$F^I_{\mathbf{X}}(\mathbf{x}) = F_1(x_1) \cdot F_2(x_2) \cdot \ldots \cdot F_n(x_n)$$

Notation: $(X_1^I, X_2^I, \ldots, X_n^I)$ for the independent random vector with marginal distributions $F_1, F_2, \ldots, F_n$.

Positive dependency

We will assume Positive Lower Orthant Dependency (PLOD):

Definition

The random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_n)$ is said to be positive lower orthant dependent (PLOD) if

$$P(X_1 \leq x_1, \cdots, X_n \leq x_n) \geq P(X_1 \leq x_1) \cdots P(X_n \leq x_n)$$

For a vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ which is PLOD, the following inequality holds for the cdf:

$$F^I_{\mathbf{X}}(\mathbf{x}) \leq F_{\mathbf{X}}(\mathbf{x}) \leq F^C_{\mathbf{X}}(\mathbf{x})$$

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Qualified Fréchet Space

Definition

The qualified Fréchet space $\mathcal{R}_n^+(F_1, \ldots, F_n)$ consists of all the n-dimensional (distribution functions $F_{\boldsymbol{X}}$ of) random vectors $\boldsymbol{X} = (X_1, \ldots, X_n)$ having $F_1, \ldots, F_n$ as marginal distribution functions and which are PLOD.

Copulas (1)

Recall:
A copula C can be seen as a function which creates a joint cdf, given any marginal cdfs.

Sklar:

$$F_{\mathbf{X}}(x_1, x_2, \ldots, x_n) = \underbrace{C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n))}_{\text{joins } F_i\text{'s AND dep. structure } C}$$

Copulas (2)

- The Independent copula:

$$C^I(u_1, u_2, \ldots, u_n) = u_1 u_2 \ldots u_n$$

- The Fréchet upper bound copula (or comonotonic copula):

$$C^U(u_1, u_2, \ldots, u_n) = min\{u_1, u_2, \ldots, u_n\}$$

- 1 parameter family copula's: parameter is linked with dependence structure
Clayton, Frank, Gumbel, ...

Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

We want: a measure that:

- is useful for all dimensions $n \geq 2$
- has values in $[0, 1]$
- $= 1$ when $\boldsymbol{X} =_d \boldsymbol{X}^c$
- $= 0$ when $\boldsymbol{X} =_d \boldsymbol{X}^I$
- holds the assumption of PLOD for $\boldsymbol{X}$

Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

Definition: Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

For all random vectors **X** of $\mathcal{R}_n^+(F_1, \ldots, F_n)$, we define the comonotonicity coefficient, $\kappa(\boldsymbol{X})$ as:

$$\kappa(\mathbf{X}) = \lim_{\substack{a \to +\infty \\ a \in \mathbb{N}}} \frac{\int \cdots \int_{G(a)} \left(F_{\mathbf{X}}(\boldsymbol{x}) - F_{\mathbf{X}}^{I}(\boldsymbol{x})\right) d\boldsymbol{x}}{\int \cdots \int_{G(a)} \left(F_{\mathbf{X}}^{C}(\boldsymbol{x}) - F_{\mathbf{X}}^{I}(\boldsymbol{x})\right) d\boldsymbol{x}},$$

provided the limit exist, and with $G(a)$ depending on the domain of $\boldsymbol{X}$ as follows:

- if **X** is defined on a finite domain, then $G(a)$ equals the whole domain

Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

Definition: Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

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provided the limit exist, and with $G(a)$ depending on the domain of $\boldsymbol{X}$ as follows:

- if **X** is defined on $\mathbb{R}^n$, then $G(a) = [-a, a]^n$

Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

Definition: Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

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provided the limit exist, and with $G(a)$ depending on the domain of $\boldsymbol{X}$ as follows:

- if **X** is defined on $\mathbb{R}^{+n}$, then $G(a) = [1/a, a]^n$

Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

Definition: Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

For all random vectors **X** of $\mathcal{R}_n^+(F_1, \ldots, F_n)$, we define the comonotonicity coefficient, $\kappa(\boldsymbol{X})$ as:

$$\kappa(\mathbf{X}) = \lim_{\substack{a \to +\infty \\ a \in \mathbb{N}}} \frac{\int \cdots \int_{G(a)} \left(F_{\mathbf{X}}(\boldsymbol{x}) - F^I_{\mathbf{X}}(\boldsymbol{x})\right) d\boldsymbol{x}}{\int \cdots \int_{G(a)} \left(F^C_{\mathbf{X}}(\boldsymbol{x}) - F^I_{\mathbf{X}}(\boldsymbol{x})\right) d\boldsymbol{x}},$$

provided the limit exist, and with $G(a)$ depending on the domain of $\boldsymbol{X}$ as follows:

- if **X** is defined on $[A_1, +\infty[\times \ldots \times [A_n, +\infty[$, then $G(a) = [A_1, a] \times \ldots \times [A_n, a]$

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Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

Definition: Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

When $F_{\mathbf{X}}(x_1, \ldots, x_n) = C_{\mathbf{X}}(F_1(x_1), \ldots, F_n(x_n))$, define the comonotonicity coefficient, $\kappa(\boldsymbol{X})$ as:

$$\kappa(\mathbf{X}) = \lim_{\substack{a \to +\infty \\ a \in \mathbb{N}}} \frac{\int \cdots \int_{H(a)} \left(C_{\mathbf{X}}(\boldsymbol{u}) - \prod_{i=1}^{n} u_i \right) dF_1^{-1}(u_1) \ldots dF_n^{-1}(u_n)}{\int \cdots \int_{H(a)} \left(\min_{i=1}^{n} u_i - \prod_{i=1}^{n} u_i \right) dF_1^{-1}(u_1) \ldots dF_n^{-1}(u_n)},$$

with $H(a) \subset [0,1]^n$ depending on the domain of **X**.

Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

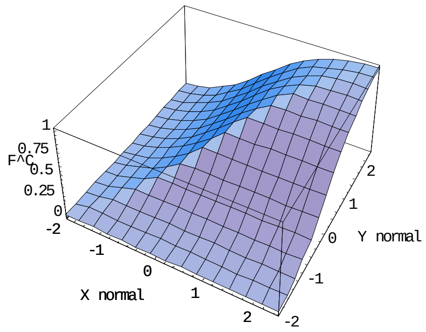
Theorem

This comonotonicity coefficient $\kappa(\mathbf{X})$ satisfies the following properties:

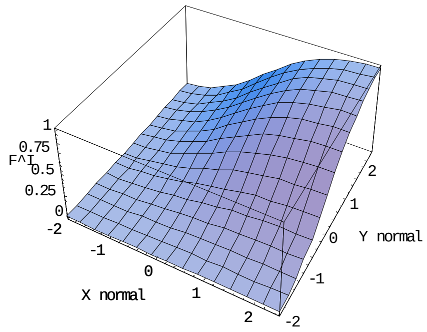
- $\kappa(\mathbf{X})$ takes values in [0,1].
- $\kappa(\boldsymbol{X})$ is symmetrical
- $\kappa(\mathbf{X}) = 1$ if and only if $(X_1, X_2, \ldots, X_n)$ has a comonotonic dependence structure.
- $\kappa(\mathbf{X}) = 0$ if and only if $(X_1, X_2, \ldots, X_n)$ is independent.

Comonotonicity Coefficient $\kappa(\boldsymbol{X})$

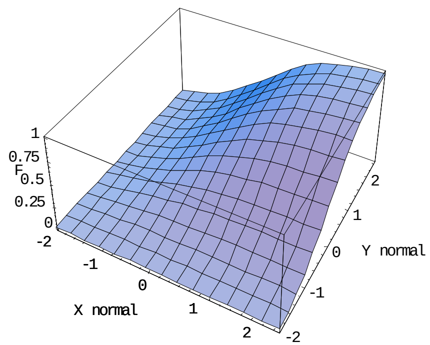
$\kappa(\mathbf{X})$ is ratio of two (hyper)volumes:



(a) joint cdf for X^C



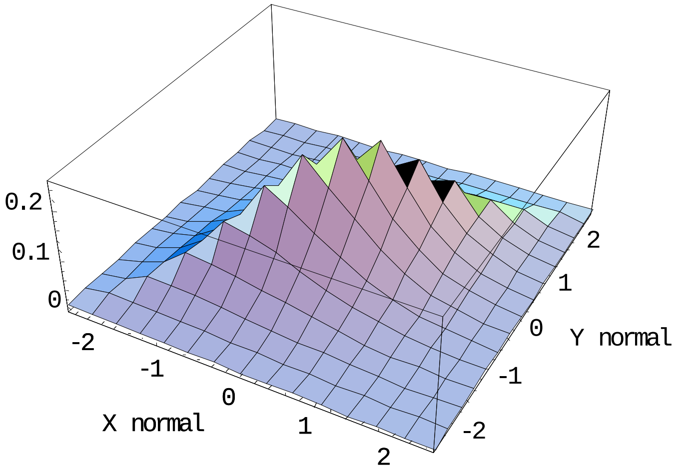
(b) joint cdf for X^I



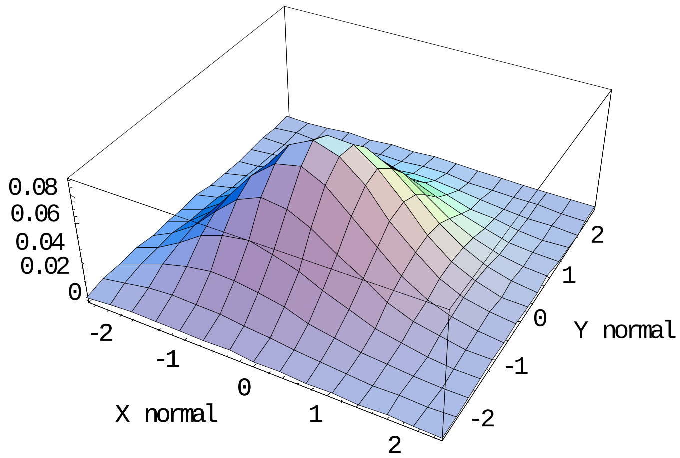
(c) joint cdf clayton(1) copula

Comonotonicity Coefficient $\kappa(X)$

$\kappa(\mathbf{X})$ is ratio of two (hyper)volumes:



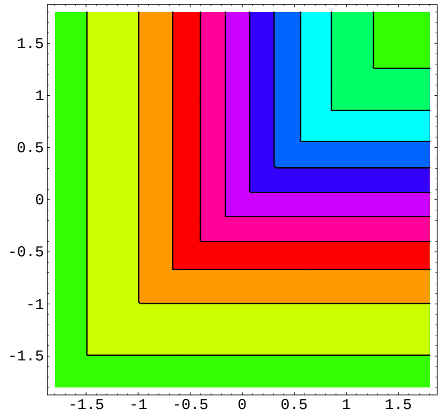
(d) volume F_X^C - F_X^I



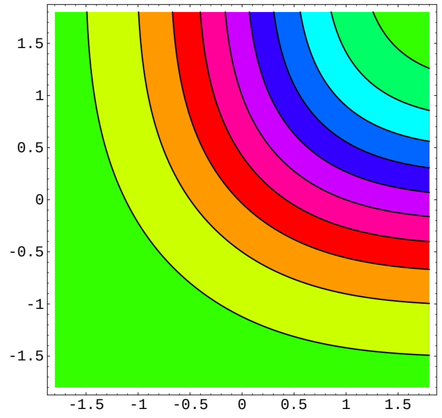
(e) volume F_X - F_X^I

$$\kappa(\mathbf{X}) = 49.8\%$$

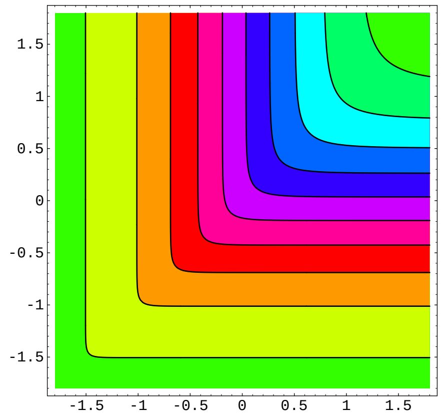
Visualization by Contourplots



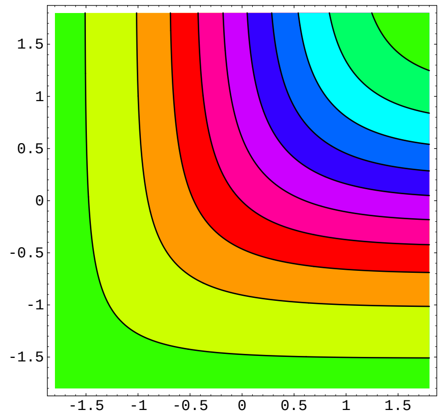
(a) cdf for X^C, $\kappa(\mathbf{X}) = 1$



(b) cdf for X^I, $\kappa(\mathbf{X}) = 0$



(c) cdf, X$\sim$clay(10),$\kappa(\mathbf{X}) = 0.935$



(d) cdf, X$\sim$clay(1),$\kappa(\mathbf{X}) = 0.498$

Properties of $\kappa(\boldsymbol{X})$ (1)

$\kappa(\mathbf{X})$ is consistent :

Gumbel copula ($\alpha \in [1, +\infty)$), n=2, standard normal						
α	1.01	1.1	1.5	2	3	4
$\kappa(\boldsymbol{X})$	0.0165	0.148	0.501	0.701	0.858	0.917
Clayton copula ($\alpha \in [0, +\infty)$), n=3, exponential						
α	0.1	1	8	10	20	50
$\kappa(\boldsymbol{X})$	0.0399	0.290	0.753	0.791	0.880	0.947
Frank copula ($\alpha \in [0, +\infty)$), n=4, uniform[0,1]						
α	0.01	1	3	5	10	15
$\kappa(\boldsymbol{X})$	0.0012	0.128	0.389	0.593	0.837	0.917

Properties of $\kappa(\boldsymbol{X})$ (2)

- Let $\rho_s(X_1, X_2)$ denote the Spearman rank correlation coefficient
- In the bivariate case, when the marginal cdfs both come from a $Uniform[0,1]$ distribution:

$$\kappa(X_1, X_2) = \rho_s(X_1, X_2)$$

- Analogue result in multivariate case

Properties of $\kappa(X)$ (3)

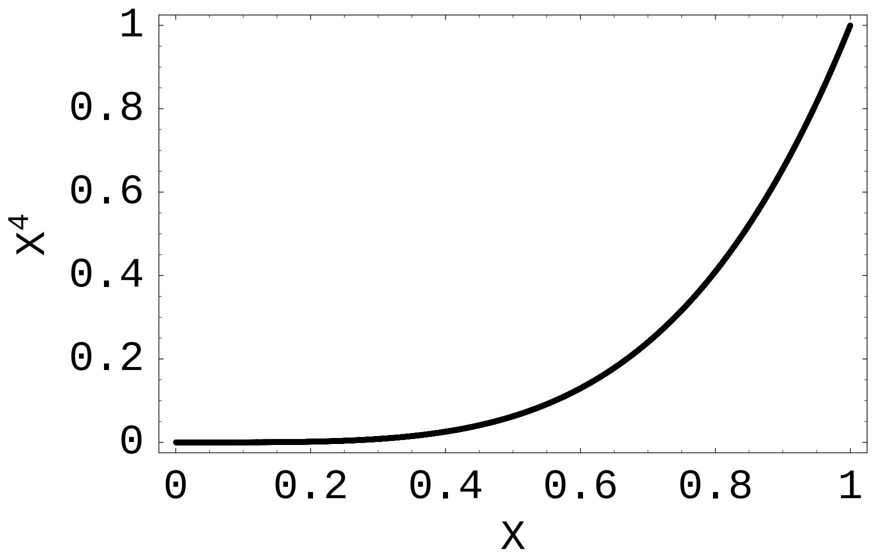
- Let $r_p(X_1, X_2)$ denote the Pearson correlation coefficient
- In the bivariate case, $\kappa(X_1, X_2)$ equals the Pearson correlation coefficient if

$$\int_0^1 \int_0^1 (\min\{u, v\} - uv)\, dF_1^{-1}(u) dF_2^{-1}(v)$$
$$= \sqrt{\mathrm{Var}(X_1)\,\mathrm{Var}(X_2)}$$

- TRUE for normal, exponentional, uniform marginals, FALSE for lognormal marginals

Properties of $\kappa(X)$ (4)

- X $\sim$ uniform[0,1], look at (X, X^4)
  Corr$(X, X^4) = 0.86507$ BUT $\kappa(X, X^4) = 1$



- In general: $\kappa(X, g_1(X), \ldots, g_n(X)) = 1$ for g_i monotonely increasing, differentiable

$\kappa(\mathbf{X})$ and other dep. measures

Franks copula, normal and lognormal marginals

α	$\kappa(\mathbf{X})$	Pearson	Spearman	Kendall
0.01	0.00167	0.00144	0.00746	0.00498
0.5	0.0830	0.0719	0.295	0.200
1	0.164	0.142	0.478	0.333
2	0.316	0.274	0.682	0.500
3	0.447	0.387	0.786	0.600
10	0.850	0.736	0.958	0.833
20	0.949	0.822	0.987	0.909

Properties of $\kappa(X)$ (5)

- If $F_{\mathbf{X}}(\mathbf{x})$ can be written as a convex sum of $F^C_{\mathbf{X}}(\mathbf{x})$ and $F^I_{\mathbf{X}}(\mathbf{x})$

$$F_{\mathbf{X}}(\mathbf{x}) = \alpha F^C_{\mathbf{X}}(\mathbf{x}) + (1-\alpha) F^I_{\mathbf{X}}(\mathbf{x}),$$

  with $\alpha \in$ [0,1]
- $\Rightarrow \kappa(\mathbf{X}) = \alpha$
- $\kappa(\mathbf{X})$ measures indeed the amount of comonotonicity in a random vector **X**
- Note that in such a case

$$\lambda_U(X_1, X_2) = \alpha = \kappa(X_1, X_2)$$

Gaussian discounting process (1)

Problem:

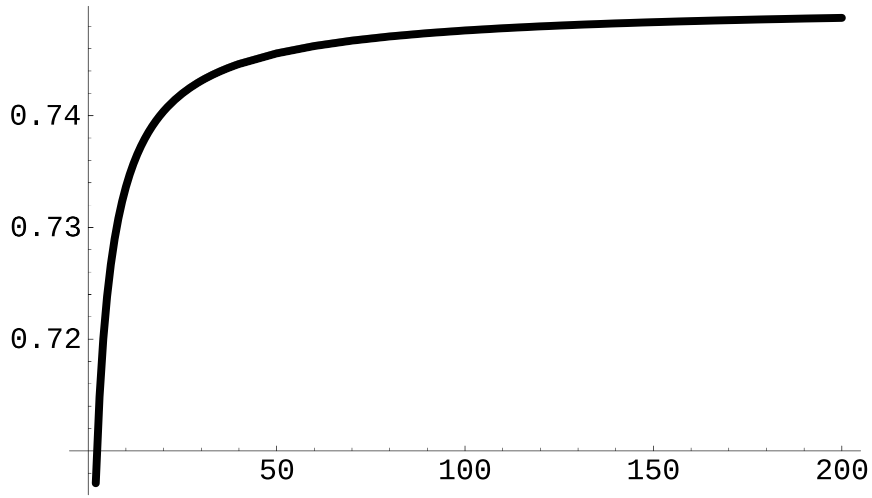
Find Present Value of Cash Flow:

$$\sum_{i=1}^{n} \alpha_i e^{-(X_1+\ldots+X_i)}$$

- there is no exact calculation
- approximate by comonotonic bounds
- seems to work $\longrightarrow$ intuition of strong dependence, no evidence
- $\Rightarrow \kappa(\mathbf{X})$!

Gaussian discounting process (2)

n	$\kappa(\mathbf{X})$	n	$\kappa(\mathbf{X})$	n	$\kappa(\mathbf{X})$
2	0.707107	6	0.726681	10	0.733621
3	0.714828	7	0.728953	100	0.747624
4	0.720031	8	0.730798	150	0.748366
5	0.723805	9	0.732329	200	0.748751



Conclusions

- $\kappa(\boldsymbol{X})$ is a coefficient which is well defined
- $\kappa(\boldsymbol{X})$ measures the amount of comonotonicity in a multivariate setting
- $\kappa(\boldsymbol{X})$ proves why the convex upper and lower bounds work so well in the case of a Gaussian discounting process
- only assumes PLOD (weak assumption of positive dependency)

Further research

- Construction of a sample-based/empirical comonotonicity coefficient $K(\boldsymbol{X})$
- Error analysis of empirical $K(\boldsymbol{X})$
- Construction of statistical test for $H_0 : \kappa(\boldsymbol{X}) = \kappa_0(\boldsymbol{X})$
- Using the value of $\kappa(\boldsymbol{X})$ to construct new approximations for the distribution function of general sums.

Contact

- Inge Koch
- University of Antwerp
- Faculty of Applied Economics
- Department of Mathematics, Statistics and Actuarial Sciences
- e-mail: inge.koch@ua.ac.be
- download paper: http://www.ua.ac.be/inge.koch