Comparison of Euclidean and prominent non-Euclidean weighted averages of covariance matrices

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Keywords: manifold, metric, positive definite, power, Procrustes, Riemannian, symmetric, smoothing, weighted Fréchet mean

As different metrics continue to be considered for measuring distances between symmetric positive semi-definite (SPD) matrices (e.g. covariance matrices), we have compared the more prominent such metrics. Our focus is on the matrix size as measured by the trace and determinant, although we also study matrix shape as measured by fractional anisotropy. For example, Diffusion Tensor Imaging (DTI), using the Euclidean distance to process covariance matrices preserves the trace and subsequenty the mean diffusivity. However, the same Euclidean approach is also often criticised for its "swelling" effect on the determinant, and for possible violation of positive definiteness in extrapolation. The affine invariant and log-Euclidean Riemannian metrics have been subsequently proposed to remedy these deficiencies. However, practitioners have also argued that these geometric approaches might be an overkill in DTI applications. We examine alternatives that in a sense reside between the Euclidean (arithmetic) and affine invariant Riemannian (geometric) extremes. These alternatives are based on the principal square root Euclidean metric and on the Procrustes size-and-shape metric. Unlike the above Riemannian metrics, these root based metrics operate more naturally (in our opinion) with regard to the boundary of the cone of SPD matrices. In particular, we prove that the Procrustes metric, when used to compute weighted averages of two SPD matrices, preserves matrix rank. We also establish and prove a key relationship between these two metrics, as well as inequalities ranking traces and determinants of weighted averages based on the Riemannian, Euclidean, and our alternative metrics. Remarkably, traces and determinants of our alternative interpolants compare differently. Experimental illustrations will also be shown. This discussion is based on [1].

References

[1] Zhou, D., Dryden, I.L., Koloydenko, A.A., Audenaert, K.M.R., Bai, L. (2016). Regularisation, interpolation and visualisation of diffusion tensor images using non-Euclidean statistics. *Journal of Applied Statistics* **43**, 943–978.