About nearly critical branching processes with dependent immigration

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Let $\{\xi_{k,\,j},\ \ k,j \in N\}$ and $\{\varepsilon_k,\ \ k \in N\}$ be independent collections of independent, nonnegative, integer-valued, identically distributed random variables. We define a sequence of random variables $\{X_k,\ \ k \in N_0\}$ by the following recurrence relations:

$$X_0 = 0,\ \ X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k, \quad k \in N.$$

The sequence $\{X_k,\ \ k \in N_0\}$ is called a branching process with immigration.

Let for each $n \in N$ $\left\{\xi_{k,j}^{(n)},\ k,j \in N\right\}$ be independent, nonnegative integer-valued random variables and $\left\{\tau_k^{(n)},\ k \in N\right\}$ be stationary process in a broad sense, the random variables $\tau_k^{(n)}$ taking nonnegative integer values. Suppose that for each $n \in N$ a set of random variables $\left\{\xi_{k,j}^{(n)},\ k,j \in N\right\}$ and the process $\left\{\tau_k^{(n)},\ k \in N\right\}$ are independent. We consider a sequence of branching processes with immigration $\left\{Z_k^{(n)},\ \ k \in N_0\right\}$, $n \in N$, following recurrence relations

$$Z_0^{(n)} = 0,\ \ Z_k^{(n)} = \sum_{j=1}^{Z_{k-1}^{(n)}} \xi_{k,j}^{(n)} + \tau_k^{(n)}, \quad k,\ n \in N.$$

We define a random step function $Z_n(t)$, $n \in N$, by setting $Z_n(t) = Z_{[nt]}^{(n)}$, $t \geq 0$, where [a] is the integer part of a. Assume that

$$m_n = E\xi_{1,1}^{(n)}\,,\ \sigma_n^2 = var\,\xi_{1,1}^{(n)},\ \gamma_n = E\tau_1^{(n)}\,,\ \delta_n^2 = var\,\tau_1^{(n)},\ \ \rho_n(k) = cov\left(\tau_1^{(n)},\ \tau_{k+1}^{(n)}\right)$$

are finite for all $n \in N$.

Theorem. Suppose that the following conditions are satisfied:
A. $m_n = 1 + \alpha n^{-1} + o(n^{-1})$ as $n \to \infty$ for any $\alpha \in R$;
B. $\sigma_n^2 \to 0$ as $n \to \infty$;
C. $\gamma_n \to \gamma \geq 0$, $\ \delta_n^2 \to \delta^2 \geq 0$ as $n \to \infty$;
D. $\frac{1}{n}\sum_{k=1}^{n} |\rho_n(k)| \to 0$ as $n \to \infty$.
Then the following weak convergence takes place in the Skorokhod space $D[0,\ \infty)$ as $n \to \infty$: $n^{-1}Z_n \to \mu$, where μ is defined by the relation $\mu(t) = \gamma\int_0^t e^{\alpha u}du$.

The case of independent, nonnegative, integer-valued, identically distributed random variables $\left\{\tau_k^{(n)},\ k \in N\right\}$ was studied in [1].

References

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