Another generalization of bivariate FGM distributions

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Let $H(x, y)$ be the bivariate cdf of (X, Y), with univariate marginals $F(x), G(y)$ and supports $[a, b]$, $[c, d]$, respectively. Throughout this abstract, x and y in $H(x, y)$, $F(x)$, $G(y)$, as well as u and v in $C(u, v)$, where $0 \leq u, v \leq 1$, will be suppressed. We write $H \in \mathcal{F}(F, G)$, where $\mathcal{F}(F, G)$ is the family of cdf's with marginals F, G.

The Farlie-Gumbel-Morgenstern (FGM) family is $H_\theta = FG[1 + \theta(1 - F)(1 - G)]$, $-1 \leq \theta \leq 1$, and the corresponding copula is $C_\theta = uv[1+\theta(1-u)(1-v)]$, $-1 \leq \theta \leq 1$. This family is frequently used in theory and applications. This motivated to study proper extensions in [2] and [1].

Let Φ, Ψ be two univariate cdf's with the same supports $[a, b]$, $[c, d]$. Suppose that the Radon-Nykodim derivatives $d\Phi/dG$, $d\Psi/dG$ exist. We define the bivariate cdf

$$H = FG + \lambda(F - \Phi)(G - \Psi).$$

This cdf reduces to the classic FGM for $\Phi = F^2, \Psi = G^2$, and has interesting properties:

1. $H \in \mathcal{F}(F, G)$ for λ belonging to an interval depending on $d\Phi/dG$, $d\Psi/dG$.
2. H suggests the congugate family $H_* \in \mathcal{F}(\Phi, \Psi)$.
3. Define $a_1 = 1 - d\Phi/dF, b_1 = 1 - d\Psi/dG$. Then $E[a_1(X)] = E[b_1(Y)] = 0$ and $E[a_1^2(X)] = \alpha - 1$, $E[b_1^2(Y)] = \beta - 1$, where $\alpha = \int_a^b (\frac{d\Phi}{dF})^2 dF$, $\quad \beta = \int_c^d (\frac{d\Psi}{dG})^2 dG$.
4. The first canonical correlation is $\rho_1 = \lambda\sqrt{(\alpha - 1)(\beta - 1)}$ and Pearson contingency coefficient is $\phi^2 = \rho_1^2$.
5. Spearman's rho and Kendall's tau are $\rho_S = 12\lambda(\frac{1}{2} - F_\Phi)(\frac{1}{2} - G_\Psi)$ and $\tau = 8\lambda(\frac{1}{2} - F_\Phi)(\frac{1}{2} - G_\Psi)$, where $F_\Phi = \int_a^b \Phi dF$, $\Phi_F = \int_c^d F d\Phi$.

The geometric dimensionality of a bivariate cdf is defined and discussed. Then we introduce the following generalized FGM

$$\begin{aligned} H &= FG + \lambda_1(F - \Phi)(G - \Psi) \\ &\quad + \lambda_2[(\tfrac{1}{2}F^2 + (F_\Phi - \tfrac{1}{2})F - F_\Phi(x)][(\tfrac{1}{2}G^2 + (G_\Psi - \tfrac{1}{2})G - G_\Psi(y)], \end{aligned}$$

where $F_\Phi(x) = \int_a^x \Phi(t)dF(t)$, $G_\Psi(y) = \int_c^y \Psi(t)dG(t)$. This $H \in \mathcal{F}(F, G)$ is diagonal and two-dimensional. Finally we study how to approximate any cdf by a member of this family.

References

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