Asymptotic bounds of the rate of convergence for some stochastic models

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Introduction.

We consider the problem of finding of asymptotic bounds for the rate of convergence to the steady state distribution for birth and death processes (BDPs) as the number of states tends to infinity. Such problems are actually investigated by a number of authors, firstly for queueing and physical applications, see for instance, [1], [2], [4], [9].

There is a well-known connection between the bounds of the spectrum of the intensity matrix of BDP, and maximal and minimal rates of convergence. An approach of estimating the minimal and maximal rates of convergence for general (homogeneous and nonhomogeneous) BDPs was suggested in [10]-[11], and applications for nonstationary Markovian queues have been studied in [3].

Let $X(t), \quad t \geq 0$ be a BDP on the state space $E = \{0, 1, \ldots, N\}$, and let $\lambda_k > 0$, $0 \leq k \leq N-1$ and $\mu_k > 0, 1 \leq k \leq N, \quad \mu_0 = \lambda_N = 0$ be birth and death intensities respectively.

Let Σ be the spectrum of the respective infinitesimal matrix. Denote by $(-\chi_N)$ and $(-\beta_N)$ the minimal and maximal points of $\Sigma \setminus \{0\}$ respectively, it is known that all points of $\Sigma \setminus \{0\}$ are real, distinct, and negative.

The present study is devoted to and estimating of χ_N, β_N and their asymptotic (as $N \to \infty$).

Consider the forward Kolmogorov system for $X(t)$ in explicit form:

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$$\begin{pmatrix} \frac{dp_0}{dt} \\ \frac{dp_1}{dt} \\ \vdots \\ \frac{dp_N}{dt} \end{pmatrix} = \begin{pmatrix} -\lambda_0 & \mu_1 & 0 & 0 & \cdots \\ \lambda_0 & -(\lambda_1+\mu_1) & \mu_2 & 0 & \cdots \\ 0 & \lambda_1 & -(\lambda_2+\mu_2) & \mu_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & \cdots & \cdots & \lambda_{N-1} & -\mu_N \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_N \end{pmatrix}, \tag{0.1}$$

where $\mathbf{p} = (p_0, p_1, \dots, p_N)^T$ is the column vector of state probabilities for $X(t)$.

Putting $p_0(t) = 1 - \sum_{i=1}^{N} p_i(t)$, we obtain the 'reduced' system

$$\frac{d\mathbf{z}(t)}{dt} = B\mathbf{z}(t) + \mathbf{f}. \tag{0.2}$$

with respective B and $\mathbf{f}$.

Let

$$\frac{d\mathbf{x}(t)}{dt} = B\mathbf{x}(t) \tag{0.3}$$

be the respective 'homogeneous' system.

Then we have for any pairs , $\mathbf{p}^{(i)} = \mathbf{p}^{(i)}(t),\ t \geq 0,\ i = 1,2$ and the respective $\mathbf{x}(t)$ the following bound in sum-norm (see, for instance, [3]):

$$\|\mathbf{x}\| \leq \left\|\mathbf{p}^{(1)} - \mathbf{p}^{(2)}\right\| \leq 2\,\|\mathbf{x}\|\,,\ t \geq 0, \tag{0.4}$$

and therefore, the rate of convergence to the stationary distribution of BDP $\boldsymbol{X(t)}$ and the rate of convergence to zero of $\mathbf{x}(\boldsymbol{t})$ are the same.

Put now

$$\boldsymbol{\alpha_k = \lambda_k + \mu_{k+1} - \delta_{k+1}\lambda_{k+1} - \delta_k^{-1}\mu_k, \quad k = 0, \ldots, N-1,} \tag{0.5}$$

and

$$\boldsymbol{\zeta_k = \lambda_k + \mu_{k+1} + \sigma_{k+1}\lambda_{k+1} + \sigma_k^{-1}\mu_k, \quad k = 0, \ldots, N-1,} \tag{0.6}$$

where $\boldsymbol{\delta_k}$ and $\boldsymbol{\sigma_k}$ are some positive numbers. Put also for convenience $\boldsymbol{\delta_0^{-1} = \delta_N = \sigma_0^{-1} = \sigma_N = 0}$.

Theorem 1. Let all birth and death intensities be positive. Then

(i) for any positive δ_k and σ_k the following bounds hold:

$$\min \alpha_k \le \beta_N \le \max \alpha_k,$$

$$\min \zeta_k \le \chi_N \le \max \zeta_k.$$

(ii) there exists a unique sequence $\{\delta_k^*\}_1^{N-1}$ such that

$$\alpha_k^* = \beta_N, \quad k = 0, \dots, N-1,$$

(iii) there exists a unique sequence $\{\sigma_k^*\}_1^{N-1}$ such that

$$\zeta_k^* = \chi_N, \quad k = 0, \dots, N-1.$$

Remark. Explicit formulae for δ_k^*, σ_k^* can be obtained for a few number of particular cases only. So, if $\lambda_k = a$, $\mu_{k+1} = b$, $k = 0, \dots, N-1$, then one can find δ_k^*, σ_k^*, see [4], and

$$\beta_N = a + b - 2\sqrt{ab}\cos\frac{\pi}{N+1} \to (\sqrt{a} - \sqrt{b})^2$$

as $N \to \infty$,

$$\chi_N = a + b + 2\sqrt{ab}\cos\frac{\pi}{N+1} \to \left(\sqrt{a} + \sqrt{b}\right)^2$$

as $N \to \infty$.

The results of Theorem can be applied for investigating the asymptotic of the bounds of the rates of convergence $\boldsymbol{\beta_N}$ and $\boldsymbol{\chi_N}$ as $\boldsymbol{N \to \infty}$, two examples of such applications are shown here.

Example 1. Queue-length process for $M/M/N/N+R$ queue.

There is a large number of investigations of the $M/M/N/N$ queue, see references in [5], [8]. Moreover, there is a number of studies of the asymptotic for the rate of convergence to stationarity as $N \to \infty$ for this queue, see for instance, [1], [9]. Bounds on the rate of convergence for general nonhomogeneous $M_t/M_t/N/N$ queue were obtained in [3].

In the general case ($R \geq 0$) the length of a queue $X(t)$ is a BDP on the state space $E = \{0, 1, \ldots, N+R\}$ with the arrival and service intensities $\lambda_{n-1} = \lambda;\ \mu_n = \mu \min(n, N), n = 1, \ldots, N+R$ respectively.

Proposition. Let $\lambda \geq 0,\ \mu > 0$. Then $\frac{\beta_{N+R}}{\mu} \to 1$, and $\frac{\chi_{N+R}}{N\mu} \to 1$ as $N \to \infty$ and for any behavior of R .

Example 2. A stochastic model of chemical reaction.

Stochastic models of chemical kinetics have been discussed in [7]. First bounds on the minimal rate of convergence have been obtained in [6].

Here we consider only the simple asymptotic for one of such models. Namely, consider the model of reaction $A + B \rightleftharpoons C$. Let $X(t)$ be the number of molecules of C. Then $X(t)$ is a BDP on the state space $E = \{0, 1, \ldots, N\}$ with birth intensities $\lambda_n = \frac{k_1}{V}(N + \gamma - n)(N - n)$ and death intensities $\mu_n = k_2 n$ respectively, where $k_1 > 0,\ k_2 > 0,\ \gamma \geq 0$ are constant, and V is the volume of the respective close compartment.

Proposition.

(i) Let $V = cN^{1+r},\ r \geq 0$. Then, as $N \to \infty$,

$$\beta_N = O(1).$$

(ii) If $r > 0$, then as $N \to \infty$,

$$\frac{\beta_N}{k_2} \to 1.$$

(iii) If $V = cN^r,\ r \geq 0$, then for sufficiently large N

$$1 < \frac{\chi_N}{\max\left(\frac{k_1}{V} N (N + \gamma),\ N k_2\right)} < 2.$$

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