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DYNAMIC CORRELATION MODELS FOR CREDIT PORTFOLIOS

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Plan of the Presentation

1) Overview of Credit Market

2) Standard Products - Index Tranches

3) Dynamic Correlation Model

4) Illustration

5) Concluding Remarks

Overview of Credit Market (Lipton 2007)

According to a recent BBA survey, by the end of 2006 the size of the market was about \$30 trillion

Main market participants:
1) banks (trading: 35% and loans: 9%)
2) hedge funds (32%)
3) insurers (8%) and others (9%)

Key credit products:
1) single name credit default swaps (CDS) (33%)
2) full index trades (30%) and index tranches (7.6%)
3) bespoke baskets (over 10 names) (12.5%) and others (16.9%)

Basket Loss Function

Let a **basket of credit names** include D_{max} individual credit default swaps (CDS-s), where each swap provides protection against a possible default of swap's reference name

Let $L(t)$ denote the **accumulated percentage loss** of the basket at valuation time t, $0 \leq L(t) \leq 1$,

Given that percentage **loss given default**, LGD, is a constant we calculate the **basket loss** as:

$$L(t) = LGD \frac{D(t)}{D_{max}}, \tag{1}$$

where $D(t)$ is the **number of defaults** occurred up to time t out of D_{max} names.

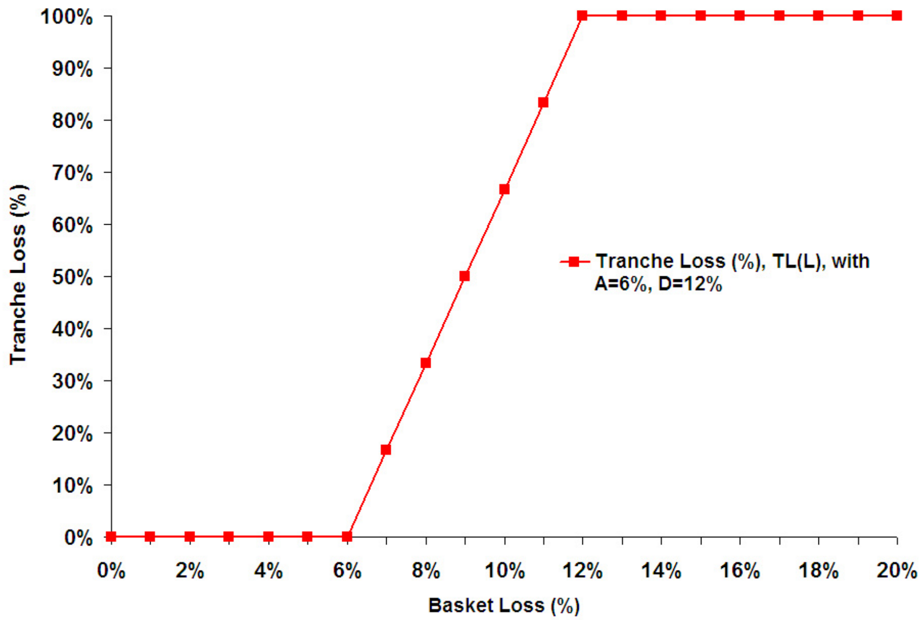
Key modeling problem: how to model $D(t)$?

Credit Tranches

Let a **credit tranche** on the basket have **attachment and detachment points** a and d, $0 \le a < d \le 1$

Let $L^{a,d}(t)$ denote the **loss function** of this tranche at time t:

$$L^{a,d}(t) = \frac{1}{d-a}\left(\min(L(t), d) - \min(L(t), a)\right). \qquad (2)$$



Credit Tranche: Premium Leg

Let us denote the annualized **payment schedule** associated with a tranche by $\{T_i\}_{i=0..N}$, with $T_0 = 0$ and $T_N = T$
Let us introduce $\Delta_i = T_i - T_{i-1}$, $i = 1..N$

The premium leg, $PL^{a,d}(T)$, of the credit tranche pays:

1) up-front payment $UF^{a,d}(T)$ paid in amount per one percent of tranche notional at contract inception

2) fixed coupon rate $S^{a,d}(T)$ at time T_i proportional to the remaining notional of the tranche at time T_i, $i = 1..N$

The expected value of its cash flows at time $t = 0$ is:

$$PL^{a,d}(T) = UF^{a,d}(T) + S^{a,d}(T) \sum_{i=1}^{N} \Delta_i DF(T_i) \mathbb{E}^{\mathbb{Q}}[1 - L^{a,d}(T_i)], \quad (3)$$

where $DF(T)$ is discount factor for risk-free cash flow at time T

Credit Tranche: Default Leg

The default leg of the tranche, $DL^{a,d}(T)$, pays at times $T_i, i = 1..N$, tranche losses experienced between $(T_{i-1}, T_i]$

The expected value of its cash flows is calculated by:

$$DL^{a,d}(T) = \sum_{i=1}^{N} DF(T_i)\mathbb{E}^{\mathbb{Q}}[L^{a,d}(T_i) - L^{a,d}(T_{i-1})]. \tag{4}$$

Fair tranche spread $S^{a,d}(T)$ equates payment and default legs:

$$S^{a,d}(T) = \frac{-UF^{a,d}(T) + \sum_{i=1}^{N} DF(T_i)\mathbb{E}^{\mathbb{Q}}[L^{a,d}(T_i) - L^{a,d}(T_{i-1})]}{\sum_{i=1}^{N} \Delta_i DF(T_i)\mathbb{E}^{\mathbb{Q}}[1 - L^{a,d}(T_i)]}. \tag{5}$$

Credit tranches are **quoted by means of their fair spreads** or up-front payments

Structured Credit Products I

Index market is now highly standardized and liquid

Key credit indices: CDX (125 US names), ITRAXX (125 European names)

Standard tranches (CDX):
$0 - 100\%$ (**full index**)
$0 - 3\%$ (**equity tranches**)
$3 - 7\%$ and $7 - 15\%$ (**mezanie tranches**)
$15 - 30\%$ (**senior tranches**)

Structured Credit Products II

Available market data (term structure of):
1) defaults probabilities of single names implied from CDS spreads
2) fair spreads of index tranches

Market for **credit structured products** is rapidly growing

Main structured products:
1) forward-start tranches
2) options on tranches
3) credit range accruals
4) super senior tranches
5) credit CPPIs and CPDOs

Dynamical Pricing Model

Static methods (copulas, entropy) have various degrees of success in fitting the market data per a single maturity

However, **pricing of structured products cannot be done within a static model** since these products depend on the evolution of implied loss surface in time

Due to **high dimensionality** (125 underlying names) pricing problem needs to be appropriately formulated

Factor credit model are originated by Duffie-Garleanu (2001), and they have been enhanced by Chapovsky *et al* (2006), Mortensen (2006), and Lipton (2006).

Key Features of Our Dynamical Model

1) Similar in spirit to Chapovsky *et al* (2006) and Lipton (2006)

2) Can be formulated in **two versions**:
i) bottom-up (consistent with default probabilities of all single names in credit basket)
ii) top-down (assuming homogeneous default probabilities in credit basket)

3) In both versions reproduces market data almost exactly across all maturities and attachment/detachment points

4) Calibration is done in closed-form

5) Pricing problem for structured credit products is formulated in PDE form

Dynamical Pricing Model

We introduce the dynamics of **market default rate** $\lambda(t)$ and **realized market default rate**, $I(t_0, t)$, under the pricing measure $\mathbb{Q}$:

$$\begin{aligned} d\lambda(t) &= \mu(t,\lambda(t))dt + \sigma(t,\lambda(t))dW(t) + J(t,\lambda(t))dN(t), \\ dI(t) &= \Upsilon(t,\lambda(t))dt,\ \lambda(0) = \lambda_0,\ I(0) = I_0, \end{aligned} \tag{6}$$

where **mapping function** $\Upsilon(t,\lambda)$ is assumed to be positive

$W(t)$ is standard **Wiener process**

$\mu(t,\lambda)$ and $\sigma(t,\lambda)$ are **drift and volatility** functions of the market default rate.

$N(t)$ is **Poisson process** with deterministic intensity $\gamma(t)$ driving the **arrival of jumps** in the market default rate.

Magnitude of jumps, $J(t,\lambda(t))$, has probability density function $\varpi(J)$

Appropriate choice of $\varpi(J)$ is **important** to fit the market data

Conditional Default Probability I

Market implied survival probability of k-th name, $Q_k^M(T)$, is implied from the term structure of CDS spreads on k-th name

We introduce **conditional survival probability** of k-th name, $Q_k(t,T)$, conditioned on realized market default rate:

$$Q_k(t,T) = \mathfrak{Q}(\beta_k(T), I(t,T)), \;\; I(t,T) = \int_t^T \Upsilon(t', \lambda(t'))dt' \text{ is given} \tag{7}$$

where $\mathfrak{Q}(\beta_k(T), I(t,T))$ is a **non-linear function** satisfying:

$$\begin{aligned} &\mathfrak{Q}(0, I(t,T)) = 1, \;\; \mathfrak{Q}(\beta, 0) = 1, \;\; \mathfrak{Q}(\beta, \infty) = 0, \\ &\overline{\mathfrak{Q}}(t,T,\beta) := \mathbb{E}^{\mathbb{Q}}\left[\mathfrak{Q}(\beta(T), I(t,T))\right] < \infty, \end{aligned} \tag{8}$$

and $\beta_k(T)$ is **the impact factor**

Conditional Default Probability II

Next we introduce the **unconditional expected survival probability** of k-th name at time t:

$$\mathfrak{G}(t,T,\beta_k(T)) = \mathbb{E}^{\mathbb{Q}}\left[\mathfrak{Q}(\beta(T),I(t,T))\right] \; \left(= \overline{\mathfrak{Q}}(t,T,\beta_k)\right) \qquad (9)$$

The **impact factor** $\beta_k(T)$ is computed in the way to equate unconditional expected survival probability to market implied default rate:

$$\mathfrak{G}(0,T,\beta_k(T)) = Q_k^M(T) \qquad (10)$$

For example, Chapovsky *et al* (2006) applied:

$$\mathfrak{Q}(\beta_k(T),I(t,T)) = e^{-\beta_k I(t,T) - \lambda_k^c(t,T) - \lambda_k^M(T)} \qquad (11)$$

Lipton (2006) employed logit survival function:

$$\mathfrak{Q}(\beta_k(T),I(t,T)) = \frac{1}{1+e^{\beta_k(T)+I(t,T)}} \qquad (12)$$

We use a similar one-parameter function

Green function of Realized Intensity Dynamics

Green function, $G^{\lambda I}(t, T, \lambda, \lambda', I, I')$, of the joint evolution of market default intensity and realized intensity solves **Kolmogoroff forward equation**:

$$
\begin{aligned}
& G^{\lambda I}_T + (\mu(T,\lambda')G^{\lambda I})_{\lambda'} - \frac{1}{2}(\sigma^2(T,\lambda')G^{\lambda I})_{\lambda'\lambda'} + (\Upsilon(T,\lambda(T))G^{\lambda I})_{I'} \\
& \quad -\gamma(T)\int_0^\infty \left(G^{\lambda I}(\lambda' - J) - G^{\lambda I}\right)\varpi(J)dJ \\
& G^{\lambda I}(t,t,\lambda,\lambda',I,I') = \delta(\lambda'-\lambda)\delta(I'-I).
\end{aligned}
\tag{13}
$$

Unconditional Green function of realized intensity, $G^I(t, T, \lambda, I, I')$, is computed by:

$$
G^I(t,T,\lambda,I,I') = \int_0^\infty G^{\lambda I}(t,T,\lambda,\lambda',I,I')d\lambda', \tag{14}
$$

Portfolio Loss Distribution at Maturity Time T

1) A grid of discrete state space, $\{I'_h\}_{h=1..H}$, of $I(t,T)$ along with corresponding probabilities, $\{P_h\}_{h=1..H}$, is constructed by solving Eq. (14) and (13)

2) Given market implied default intensities $\{\lambda_k^M(T)\}_{k=1..D_{max}}$ and the discretisized distribution of $I(t,T)$, equation (9) is solved for each name k, $k = 1..D_{max}$, to obtain $\{\beta_k(T)\}_{k=1..D_{max}}$

3) Since given a realization of $I(t,T)$, the default probabilities of individual names are independent among each other, for each state I'_h, $h = 1..H$, portfolio default distribution is non-homogeneous Binomial distribution with survival probabilities given by $\{\mathfrak{Q}(\beta_k(T), I(t,T))\}_{k=1..D_{max}}$

4) The distribution of portfolio losses is obtained by computing the average of the portfolio loss distributions obtained in 3) weighted by the probability of the corresponding state obtained in 1)

Calibration of Our Model to CDX IG 8 index Data, May 2007

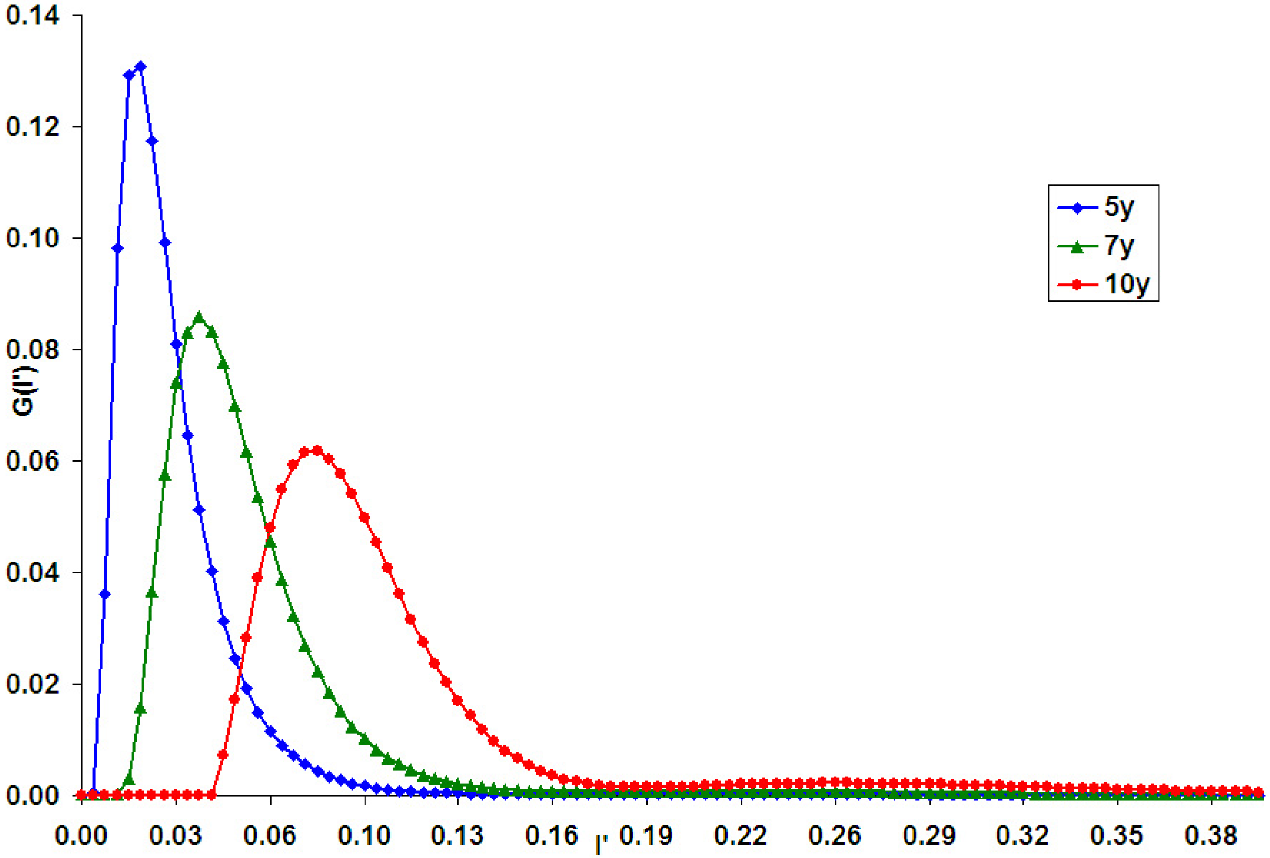
`First part:` **market implied expected tranche losses** (in %)
`Second part:` **market quotes for full index and index tranches**

Tranche	5y	7y	10y	5y	7y	10y
0-100%	1.9300%	3.5800%	6.5800%	36.00	48.00	62.00
0-3%	52.3700%	77.8600%	96.6200%	24.94%	41.19%	52.06%
3-7%	5.2600%	17.1500%	50.6200%	99.00	228.00	493.00
7-10%	1.1600%	3.8500%	14.3300%	21.50	49.50	125.50
10-15%	0.5100%	1.8800%	6.8500%	9.50	24.00	59.00
15-30%	0.2100%	0.7200%	2.2200%	3.88	9.19	19.13

`First part:` **model expected tranche losses** (in %)
`Second part:` **differences between the market and model expected tranche losses** (in %)

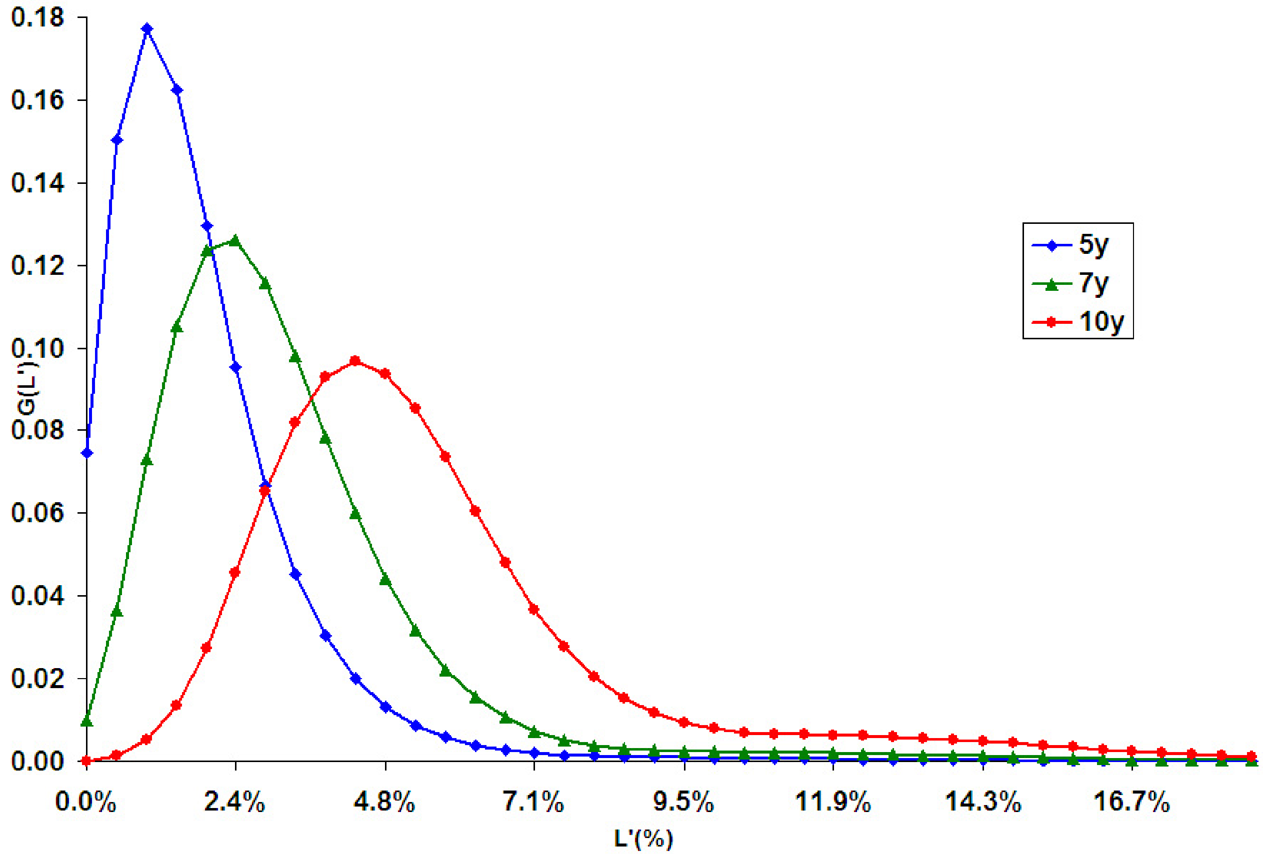
Tranche	5y	7y	10y	5y	7y	10y
0-100%	1.9324%	3.5459%	6.5877%	0.0024%	-0.0341%	0.0077%
0-3%	52.3699%	77.8610%	96.6190%	-0.0001%	0.0010%	-0.0010%
3-7%	5.2599%	17.1512%	50.6199%	-0.0001%	0.0012%	-0.0001%
7-10%	1.1599%	3.8516%	14.3295%	-0.0001%	0.0016%	-0.0005%
10-15%	0.5100%	1.8801%	6.8501%	0.0000%	0.0001%	0.0001%
15-30%	0.2088%	0.7361%	2.2163%	-0.0012%	0.0161%	-0.0037%

Implied Distribution of Realized Intensity



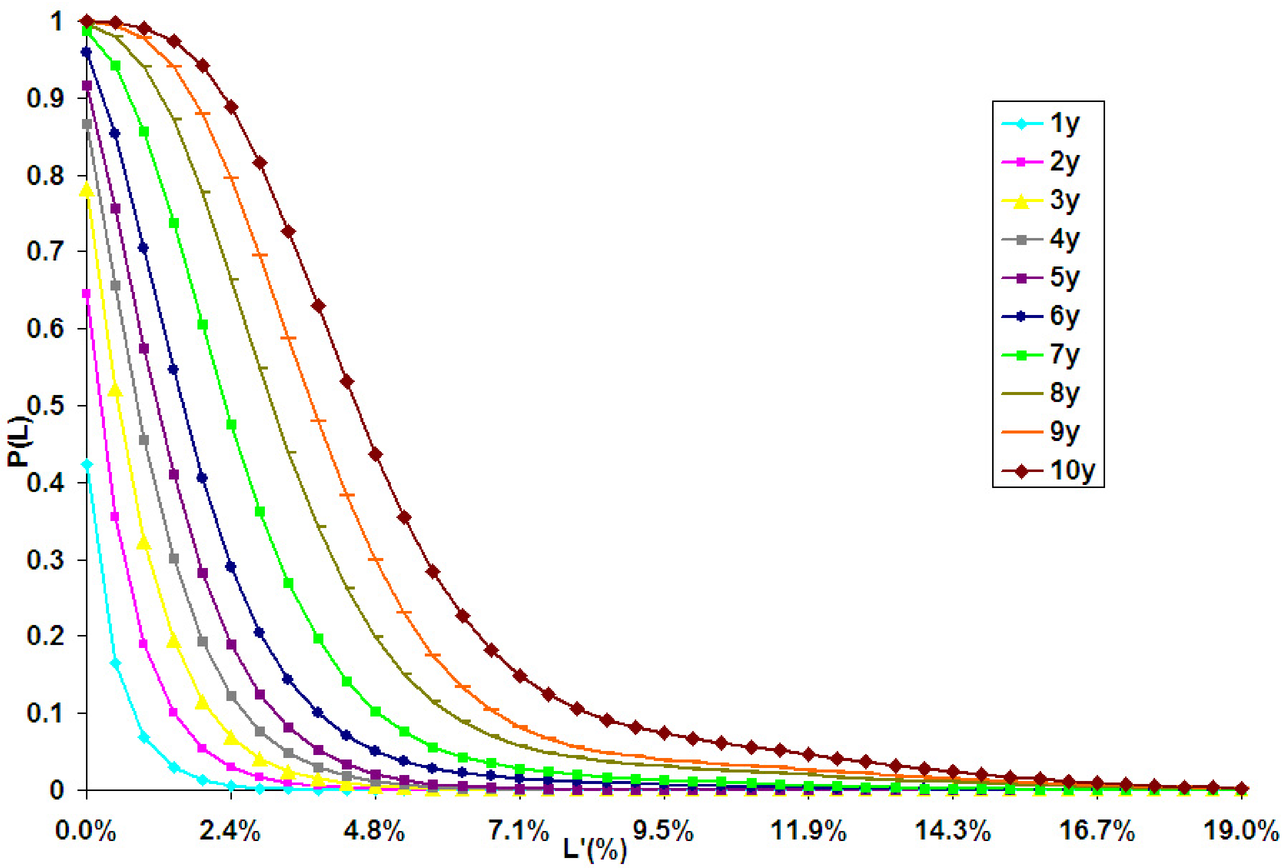
At 10y maturity, the model implies a heavy right tail to fit implied losses in mezzanine and senior tranches.

Implied loss distributions



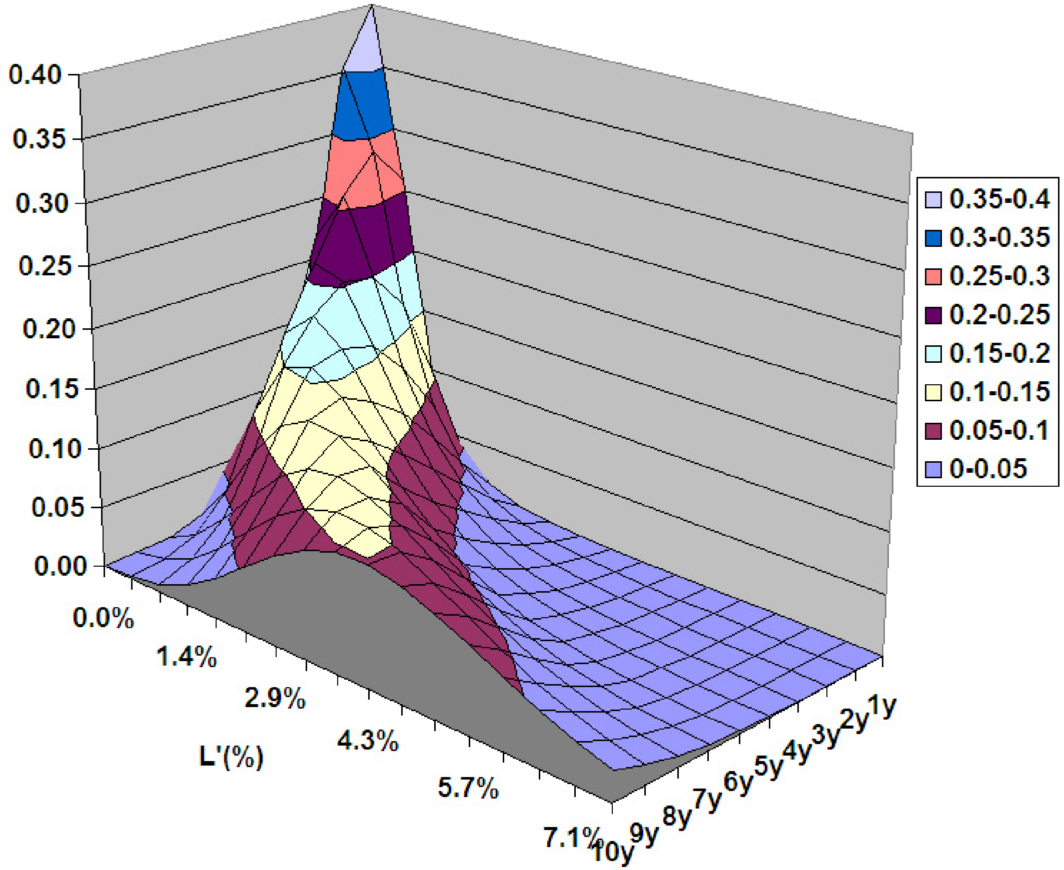
L' stands for percentage loss.

Implied Cumulative Loss Distributions



Cumulative loss distribution: $\mathbb{P}^{\mathbb{Q}}[L(t) > L']$
Our model admits no arbitrage in maturity dimension

The Density of Implied Loss Surface



Our model produces smooth arbitrage-free loss distribution both in strike and maturity dimension consistently with market data

Pricing of Structured Credit Products in PDE formulation

In general, we need to solve **backward Kolmogoroff equation** for value function, $U(t,T,\lambda,I,D)$, of a credit product with:
1) payoff function $u_1(\lambda,I,D)$ at maturity time T
2) reward function $u_2(t,\lambda,I,D)$ at time t, $0<t<T$

$$\begin{aligned}
U_t &+ \mu(t,\lambda)U_\lambda + \frac{1}{2}\sigma^2(t,\lambda)U_{\lambda\lambda} + \Sigma(D)\Upsilon(t,\lambda)U_{I'} \\
&+ \gamma(t)\int_0^\infty \left(U(\lambda+J)-U\right)\varpi(J)dJ \\
&+ \sum_{\Delta D=1}^{D_{max}-D} \Lambda(t,\lambda,I,D,\Delta D)(U(D+\Delta D)-U) \\
&- r(t)U = -u_2(t,\lambda,I,D), \\
U(T,T,\lambda,I,D) &= u_1(\lambda,I,D)
\end{aligned} \tag{15}$$

where $r(t)$ is deterministic interest rate
$\Sigma(D)=\sum_{k=D}^{D_{max}}\frac{\partial}{\partial I}\mathfrak{G}(t,T,\beta_k(T))$
$\Lambda(t,\lambda,I,D,\Delta D)$ is **loss transition rate**

Loss Transition Rate

$\Lambda(t, \lambda, I, D, \Delta D)$ is **auxiliary function** which describes arrival of ΔD defaults, $\Delta D = 1, .., D_{max} - D$, during infinitesimal time interval $[t, t + \delta t]$ given the state of the dynamics at time t

Two ways to specify $\Lambda(t, \lambda, I, D, \Delta D)$:

1) bottom-up approach: implicit specification by state variables and model parameters

2) top-down approach: explicit specification by assuming homogeneous default probabilities

In our formulation, the latter specification results in **aggregated dynamic correlation model**, which is:

1) computationally simpler than the full model

2) reproduces market data as exactly as the full model does