Copulæ measure–preserving transformations and compatibility

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Outline

C–volume

A n–*box* is a cartesian product

$$[\mathbf{a}, \mathbf{b}] = \prod_{j=1}^{n} [a_j, b_j],$$

where, for each index $j \in \{1, 2, \dots, n\}$, $0 \le a_j \le b_j \le 1$.
For a function $C : [0,1]^n \to [0,1]$, the C–volume V_C of $[\mathbf{a}, \mathbf{b}]$ is defined by

$$V_C([\mathbf{a}, \mathbf{b}]) := \sum_{\mathbf{v}} \operatorname{sign}(\mathbf{v})\, C(\mathbf{v})$$

where the sum is taken over the 2^n vertices $\mathbf{v}$ of the box $[\mathbf{a}, \mathbf{b}]$; here

$$\operatorname{sign}(\mathbf{v}) = \begin{cases} 1, & \text{if } v_j = a_j \text{ for an even number of indices,} \\ -1, & \text{if } v_j = a_j \text{ for an odd number of indices.} \end{cases}$$

What is a copula?

A function $C_n : [0,1]^n \to [0,1]$ is an n–copula if

- $C_n(x_1, x_2, \dots, x_n) = 0$ if $x_j = 0$ for at least one index $j \in \{1, 2, \dots, n\}$
- $C_n(1, 1, \dots, 1, x_j, 1, \dots, 1) = x_j$
- the V_C–volume of every n–*box* $[\mathbf{a}, \mathbf{b}]$ is positive $V_C([\mathbf{a}, \mathbf{b}]) \geq 0$.

The set of n–copulas ($n \geq 3$) is denoted by $\mathcal{C}_n$, while the set of (bivariate) copulas is denoted by $\mathcal{C}$.

What is a copula?–2

If $n = 2$ a *copula* C satisfies

- $\forall s \in [0,1]$ $t \mapsto C(s,t)$ and $t \mapsto C(t,s)$ are increasing;
- The Lipschitz condition holds

$$|C(s',t') - C(s,t)| \leq |s' - s| + |t' - t|$$

  $s \leq s'$, $t \leq t'$.
- For every copula C: $W \leq C \leq M$, where

$$W(s,t) := (s+t-1) \vee 0 \qquad M(s,t) := s \wedge t.$$

marginals

A *marginal* of an n–copula C is obtained by setting some of its arguments equal to 1. An m–marginal of C, $m < n$, is obtained by setting exactly $n - m$ arguments equal to 1; there are

$$\binom{n}{m}$$

m–marginals.

Measure–preserving transformations

$(\Omega, \mathcal{F}, \mu)$ and $(\Omega', \mathcal{F}', \nu)$ — two measure spaces.
$f : \Omega \to \Omega'$ is a measure–preserving transformation (=mpt) if

- $\forall B \in \mathcal{F}' \quad f^{-1}(B) \in \mathcal{F}$
- $\forall B \in \mathcal{F}' \quad \mu\left(f^{-1}(B)\right) = \nu(B)$

From now on $(\Omega, \mathcal{F}, \mu) = (\Omega', \mathcal{F}', \nu) = ([0,1], \mathcal{B}([0,1]), \lambda)$
$\mathcal{B}([0,1])$ — the Borel sets of $[0,1]$
λ — the (restriction of) Lebesgue measure to $\mathcal{B}([0,1])$.

Copulæ and mpt's

Theorem

If f_1, f_2, $\dots f_n$ are measure–preserving transformations, then the function $C_{f_1,f_2,\dots,f_n} : [0,1]^n \to [0,1]$ defined by

$$C_{f_1,f_2,\dots,f_n}(x_1, x_2, \dots, x_n) := \lambda\left(f_1^{-1}[0,x_1] \cap \dots \cap f_n^{-1}[0,x_n]\right) \quad (1)$$

is an n–copula. Conversely, for every n–copula C, there exist n measure–preserving transformations f_1, f_2, $\dots f_n$ such that

$$C = C_{f_1,f_2,\dots,f_n}. \quad (2)$$

The representation of eq. (1) is not unique: if φ is another mpt on $[0,1]$, then one has

$$C_{f_1,f_2,\dots,f_n} = C_{f_1\circ\varphi,f_2\circ\varphi,\dots,f_n\circ\varphi}.$$

Two consequences

Corollary

If f is strongly mixing, then, for all x and y in $[0,1]$,

$$\lim_{n\to+\infty} C_{f^n,g}(x,y) = xy = \Pi(x,y). \tag{3}$$

Corollary

If f is ergodic, then, for all x and y in $[0,1]$,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} C_{f^j,g}(x,y) = xy = \Pi(x,y). \tag{4}$$

An example

$f(x) := [2x]$ where $[t]$ denotes the fractional part of t and let g be the identity function, $g(t) := t$. Then f is strongly mixing, $f^n(x) = [2^n x]$ and, for every $x \in [0,1]$ and every $y \in [(m-1)/2^n, m/2^n]$,

$$C_{f^n,g}(x,y) = x\,\frac{m-1}{2^n} + \min\left\{\frac{x}{2^n}, y - \frac{m-1}{2^n}\right\}.$$

Thus $(C_{f^n,g})_{n\in\mathbb{N}}$ is a strictly increasing sequence of copulas with

$$0 \le \Pi(x,y) - C_{f^n,g}(x,y) < \frac{1}{2^n}$$

for every point $(x,y) \in [0,1]^2$.

More examples

For the copula M one has

$$\lambda\left(f^{-1}[0,x]\cap f^{-1}[0,y]\right)=\lambda\left(f^{-1}([0,x]\cap[0,y])\right)$$
$$=\lambda([0,x]\cap[0,y])=\min\{x,y\}=M(x,y).$$

for every measure–preserving transformation.
W concentrates all the probability mass uniformly on the the diagonal $\varphi(t)=1-t$ of the unit square. In this case $\varphi=\varphi^{-1}$, and

$$W(x,y)=\lambda\left(\varphi^{-1}[0,x]\cap[0,y]\right).$$

Shuffles of Min

The probability mass of the copula M is spread uniformly on the main diagonal $f(t) = t$ of the unit square. A shuffle of M is obtained by dividing the interval $[0,1]$ into a finite number of interval $\{J_1, J_2, \dots, J_n\}$ having at most an end point in common, by permuting (shuffling) the strips $J_k \times [0,1]$, and, possibly, by flipping some of them around their vertical axes of symmetry, and, finally, by reassembling them to form the unit square again. Formally a shuffle of M is obtained by choosing a natural number n, n intervals $\{J_1, J_2, \dots, J_n\}$, a permutation π on $\{1, 2, \dots, n\}$ a function $\sigma : \{1, 2, \dots, n\} \to \{-1, 1\}$, where $\sigma(k) = -1$ or 1 according to whether the strip $J_k \times [0,1]$ is flipped or not.

Shuffles of Min–2

Theorem

Let the copula C be a shuffle of M and let φ be the equation of the piece–wise linear curve that supports the probability mass. Then, for all $(x, y) \in [0, 1]^2$,

$$C(x, y) = \lambda\left(\varphi^{-1}[0, x] \cap [0, y]\right).$$

The formulation

The general problem: Let $\binom{n}{m}$ m–copulæ be given, does there exist an n–copula C_n of which the given m–copulæ are the m–marginals?

In general the answer is No.

If $n = 3$ and $m = 2$ and the three two copulæ are all equal to W, then there is no 3–copula C of which they are the marginals. In fact if two random variables X and Y have W as their copula, then each of them is a decreasing function of the other one; but three random variables cannot be each a decreasing function of the remaining two.
If an n–copula C_n exists of which the given copulæ are the m–marginals, then these are said to be compatible.

The special case $n = 3$, $m = 2$

For $A, B \in \mathcal{C}_2$, denote by $\mathcal{D}(A, B)$ the set of all bivariate copulas that are compatible with A and B, in the sense that, if C is in $\mathcal{D}(A, B)$, then there exists a $\widetilde{C} \in \mathcal{C}_3$ such that, for all $(x, y, z) \in [0, 1]^3$

$$\widetilde{C}(x, y, 1) = A(x, y), \qquad \widetilde{C}(1, y, z) = B(y, z),$$
$$\widetilde{C}(x, 1, z) = C(x, z).$$

$\mathcal{D}(A, B) \neq \emptyset$ (Joe, 1997)

The $*$-operation

For $A, B \in \mathcal{C}_2$ the $*$–operation is a binary operation on $\mathcal{C}_2$; it is defined, for all $(x, y) \in [0, 1]^2$, by

$$(A * B)(x, y) := \int_0^1 D_2 A(x, t)\, D_1 B(t, y)\, dt,$$

where, for a copula C,

$$D_1 C(x, y) := \frac{\partial C(x, y)}{\partial x} \quad \text{and} \quad D_2 C(x, y) := \frac{\partial C(x, y)}{\partial y};$$

these partial derivatives exist a.e. on the interval $[0, 1]$ with respect to Lebesgue measure and, where they exist, are bounded below by 0 and above by 1.
(Darsow, Nguyen, Olsen, 1991)

A "new" operation

Theorem

Given two copulas $A, B \in \mathcal{C}_2$*, the function* $C_{A,B} : [0,1]^3 \to [0,1]$ *defined by*

$$C_{A,B}(x,y,z) := \int_0^y D_2 A(x,t)\, D_1 B(t,z)\, dt;$$

is a 3–copula, $C_{A,B} \in \mathcal{C}_3$*, whose marginals are given by*

$$C_{A,B}(x,y,1) = A(x,y), \qquad C_{A,B}(1,y,z) = B(y,z),$$
$$C_{A,B}(x,1,z) = (A * B)(x,z).$$

As a consequence $\mathcal{D}(A,B) \neq \emptyset$

Examples

$$
\begin{aligned}
C_{W,W}(x,y,z) &= \max\{0, y + (x \wedge z) - 1\}, \\
C_{M,M}(x,y,z) &= x \wedge y \wedge z = M_2(x,y,z), \\
C_{W,M}(x,y,z) &= \max\{0, x + (y \wedge z) - 1\}, \\
C_{M,W}(x,y,z) &= \max\{0, (x \wedge y) - 1 + z\}, \\
C_{\Pi,\Pi}(x,y,z) &= xyz = \Pi_2(x,y,z), \\
C_{\Pi,M}(x,y,z) &= x\, M(y,z), \\
C_{M,\Pi}(x,y,z) &= z\, M(x,y), \\
C_{\Pi,A}(x,y,z) &= x\, A(y,z), \\
C_{A,\Pi}(x,y,z) &= z\, A(x,y), \\
C_{W,A}(x,y,z) &= \max\{0, A(y,z) - A(1-x,z)\}.
\end{aligned}
$$

Properties of $\mathcal{D}(A, B)$

The set $\mathcal{D}(A, B)$ of copulas that are compatible with two given bivariate copulas A and B is

- convex
- compact with respect to the topology of uniform convergence in $[0, 1]^2$

MPT's and minimality of $\mathcal{D}(A, B)$

When $\mathcal{D}(A, B) = \{A * B\}$? namely when is $\mathcal{D}(A, B)$ minimal?

Theorem

Let A and B be two bivariate copulas with $A = C_{f,g}$ and $B = C_{p,r}$, where f, g, p and r are measure–preserving transformations from $[0, 1]$ into $[0, 1]$, and either f and g or p and r are one–to–one. Then $\mathcal{D}(A, B)$ is minimal.

Some open problems

1. while determining the copula $C_{(f,g)}$ associated with a pair of measure–preserving transformations (f, g) is easy, the converse problem of finding such a pair for a given copula C is considerably harder; for instance, given the copula Π, which is the pair (f, g) such that $\Pi = C_{(f,g)}$?
2. to characterize the set $D(A) := D(A, A)$ for a given 2–copula A. Notice that $D(\Pi) = \mathcal{C}_2$, since for every copula A one has, if $C(x, y, z) = y\, A(x, z)$, $C(x, y, 1) = xy = \Pi(x, y)$, $C(1, y, z) = yz = \Pi(y, z)$ and $C(x, 1, z) = A(x, z)$.
3. to characterize the class $\mathcal{C}(A)$ of pairs (B, C) of 2–copulas such that A, B, and C are compatible.