On constructions and boundaries for some type of 2-increasing functions

Susanne Saminger[1,2]

[1]Department of Knowledge-Based Mathematical Systems
Johannes Kepler University Linz, Austria

susanne.saminger@jku.at

[2]Dipartimento di Matematica "Ennio De Giorgi"
Università del Salento, Lecce, Italy

susanne.saminger@unile.it

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Outline

Binary 2-increasing aggregation operator

(Durante et al., 2007)

Definition

A function $A\colon [0,1]^2 \to [0,1]$ is called a binary aggregation operator if

- $A(0,0) = 0,\ A(1,1) = 1;$
- $A(x,y) \leq A(x',y')$, whenever $x \leq x'$, $y \leq y'$.

Binary 2-increasing aggregation operator

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Definition

A function $A\colon \mathbb{R}^2 \to \mathbb{R}$ is called 2-increasing (supermodular) if, for all $[x_1,x_2] \times [y_1,y_2] \subset \mathbb{R}^2$ with $x_1 \leq x_2$ and $y_1 \leq y_2$,

$$V_A([x_1,x_2] \times [y_1,y_2]) := A(x_1,y_1) + A(y_1,y_2) - A(x_1,y_2) - A(x_2,y_1) \geq 0.$$

Binary 2-increasing aggregation operator

(Durante et al., 2007)

Proposition

Consider a 2-increasing binary aggregation operator A.

- Then, for every $f, g\colon [0,1] \to [0,1]$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$, the function $A_{f,g}\colon [0,1]^2 \to [0,1]$, given by

$$A_{f,g}(x,y) = A(f(x), g(y))$$

is a 2-increasing binary aggregation operator.

Binary 2-increasing aggregation operator

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$$A_{f,g}(x,y) = A(f(x), g(y))$$

  is a 2-increasing binary aggregation operator.

- Then, for given $f\colon [0,1] \to [0,1]$ with $f(0) = 0$ and $f(1) = 1$, the function $f \circ A\colon [0,1]^2 \to [0,1]$, given by

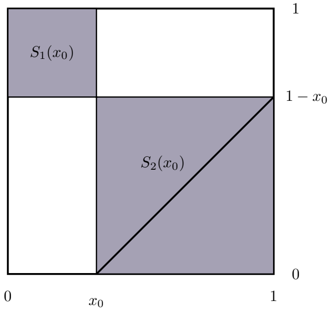
$$f \circ A(x,y) = f(A(x,y))$$

  is a 2-increasing binary aggregation operator if and only if f is convex and increasing.

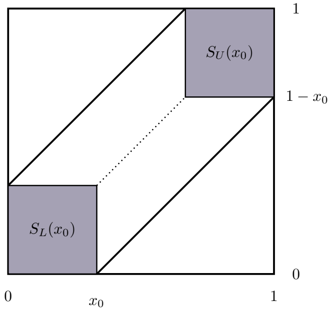
(Quasi-)copulas with given subdiagonal section

(Quesada-Molina et al., 2007)

W-ordinal sums



Symmetric copulas with given sub-diagonal section

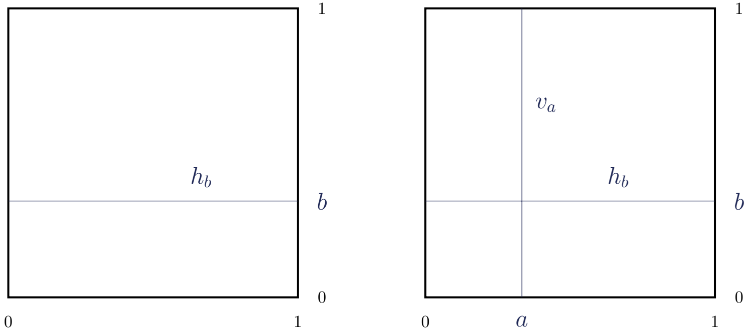


Copulas with given horizontal and/or vertical sections

(Durante et al., 2007; Klement et al., 2007)

Copulas with given ....

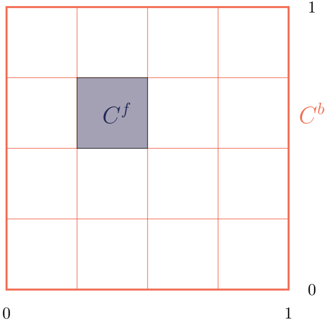
horizontal section | horizontal and vertical section



Copulas constructed on an orthogonal grid

(De Baets et al., 2007)

Copulas based on some foreground and background copula



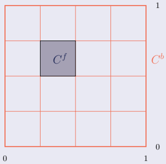
Copulas constructed on an orthogonal grid

(De Baets et al., 2007)

Theorem

Consider

- *two copulas* C^b *and* C^f,
- *a rectangle* $[a_1, a_2] \times [b_1, b_2] \subseteq [0,1]^2$, *and*
- *a positive constant* μ.



Then $A\colon [0,1]^2 \to [0,1]$, *defined, for all* $x, y \in [0,1]$, *by*

$$A(x,y) = \begin{cases} C^b(x,y) + \mu\left(C^f\left(\frac{x-a_1}{a_2-a_1}, \frac{y-b_1}{b_2-b_1}\right) - C^b\left(\frac{x-a_1}{a_2-a_1}, \frac{y-b_1}{b_2-b_1}\right)\right), & \text{if } (x,y) \in [a_1,a_2] \times [b_1,b_2], \\ C^b(x,y), & \text{otherwise}, \end{cases}$$

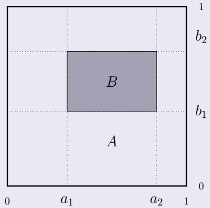
is a copula whenever $(x,y) \mapsto C^b(x,y) - \mu C^b\left(\frac{x-a_1}{a_2-a_1}, \frac{y-b_1}{b_2-b_1}\right)$ *is increasing and 2-increasing on* $[a_1,a_2] \times [b_1,b_2]$.

Problem statement

Rectangular patchwork

Consider

- a rectangle $R = [a_1, a_2] \times [b_1, b_2] \subseteq [0, 1]^2$ with $a_1 < a_2$, $b_1 < b_2$;
- a binary function $A \colon [0, 1]^2 \to [0, 1]$ which is also 2-increasing;
- a binary function $B \colon R \to [0, 1]$.



Then the function $A \square_R B \colon [0, 1]^2 \to [0, 1]$ defined by

$$A \square_R B(x, y) = \begin{cases} B(x, y), & \text{if } (x, y) \in R, \\ A(x, y), & \text{otherwise,} \end{cases}$$

is called rectangular patchwork of A and B on $[0, 1]^2$.

Problem statement

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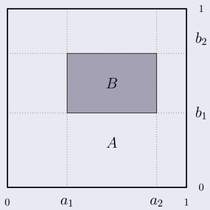
Consider

- a rectangle $R = [a_1, a_2] \times [b_1, b_2] \subseteq [0, 1]^2$ with $a_1 < a_2$, $b_1 < b_2$;
- a binary function $A\colon [0, 1]^2 \to [0, 1]$ which is also 2-increasing;
- a binary function $B\colon R \to [0, 1]$ such that $B(a_i, y) = A(a_i, y)$ for all $y \in [b_1, b_2]$ and $B(x, b_i) = A(x, b_i)$ for all $x \in [a_1, a_2]$.



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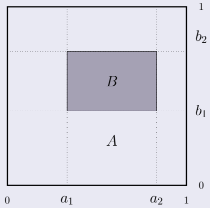


For $A\square_R B$ to be 2-increasing:

- What are the possible choices for B?
- What are the largest and smallest possible B's?
- How can such B be represented?

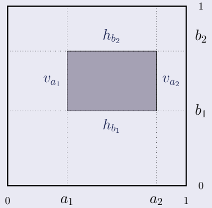
Set of margins

Definition

Consider arbitrary $a_1, a_2, b_1, b_2 \in [0,1]$ with $a_1 < a_2$ and $b_1 < b_2$.

A set $M = \{h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2}\}$ of four increasing functions $h_{b_i}\colon [a_1, a_2] \to [0,1]$, $v_{a_i}\colon [b_1, b_2] \to [0,1]$, $i = 1,2$, is called a set of margins if the following conditions are fulfilled:

(M1) $h_{b_1}(a_1) = v_{a_1}(b_1),\ h_{b_1}(a_2) = v_{a_2}(b_1),$
$h_{b_2}(a_1) = v_{a_1}(b_2),\ h_{b_2}(a_2) = v_{a_2}(b_2).$



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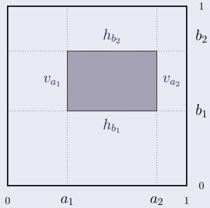
$h_{b_2}(a_1) = v_{a_1}(b_2)$, $h_{b_2}(a_2) = v_{a_2}(b_2)$.

(M2) For all $x_1, x_2 \in [a_1, a_2]$ and all $y_1, y_2 \in [b_1, b_2]$ with $x_1 \le x_2$, $y_1 \le y_2$:

$$h_{b_2}(x_2) + h_{b_1}(x_1) \ge h_{b_2}(x_1) + h_{b_1}(x_2);$$
$$v_{a_2}(y_2) + v_{a_1}(y_1) \ge v_{a_2}(y_1) + v_{a_1}(y_2).$$



Set of margins

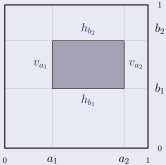
Properties of margins

Consider

- arbitraray $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$ and $b_1 < b_2$ and
- a set of margins $M = \{h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2}\}$.

Then for all $x, x' \in [a_1, a_2]$ and for all $y, y' \in [b_1, b_2]$ with $x' \geq x$ and $y' \geq y$:

$$h_{b_2}(x') - h_{b_1}(x') \geq h_{b_2}(x) - h_{b_1}(x);$$
$$v_{a_2}(y') - v_{a_1}(y') \geq v_{a_2}(y) - v_{a_1}(y).$$



Set of margins

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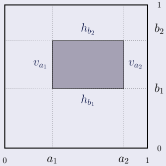
Then for all $x, x' \in [a_1, a_2]$ and for all $y, y' \in [b_1, b_2]$ with $x' \geq x$ and $y' \geq y$:



$$
\begin{aligned}
h_{b_2}(x') - h_{b_1}(x') &\geq h_{b_2}(x) - h_{b_1}(x); \\
v_{a_2}(y') - v_{a_1}(y') &\geq v_{a_2}(y) - v_{a_1}(y).
\end{aligned}
$$

$$
\begin{aligned}
h_{b_2}(x) &\geq h_{b_1}(x) + h_{b_2}(a_1) - h_{b_1}(a_1) \\
&\qquad = h_{b_1}(x) + v_{a_1}(b_2) - v_{a_1}(b_1) \geq h_{b_1}(x); \\
v_{a_2}(y) &\geq v_{a_1}(y) + v_{a_2}(b_1) - v_{a_1}(b_1) \\
&\qquad = v_{a_1}(y) + h_{b_1}(a_2) - h_{b_1}(a_1) \geq v_{a_1}(y).
\end{aligned}
$$

Set of margins

Problemstatement revisited

Consider

- arbitraray $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$ and $b_1 < b_2$ and
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Set of margins

Problemstatement revisited

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- arbitraray $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$ and $b_1 < b_2$ and
- a set of margins $M = \{h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2}\}$.

We will denote by $\mathcal{M}_2$ the set of all increasing 2-increasing functions $A\colon [a_1, a_2] \times [b_1, b_2] \to [0,1]$ coinciding in its set of margins with M.

Set of margins

Problemstatement revisited

Consider

- arbitraray $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$ and $b_1 < b_2$ and
- a set of margins $M = \{h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2}\}$.

We will denote by $\mathcal{M}_2$ the set of all increasing 2-increasing functions $A\colon [a_1, a_2] \times [b_1, b_2] \to [0,1]$ coinciding in its set of margins with M.

- Is $\mathcal{M}_2 = \emptyset$?
- What are its bounds?
- How can it be represented?

Special case: $R = [0,1]^2$, $\mathrm{Ran}_A = [0,1]$

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Representation

For every 2-increasing binary aggregation operator $A\colon [0,1]^2 \to [0,1]$ with margins h_0^A, h_1^A, v_0^A, and v_1^A, there exists a copula C such that, for all $x, y \in [0,1]$,

$$A(x,y) = C(h_1^A(x), v_1^A(y)).$$

Special case: $R = [0,1]^2$, $\mathrm{Ran}_A = [0,1]$

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$$A(x,y) = C(h_1^A(x), v_1^A(y)).$$

C is uniquely determined on $\mathrm{Ran}_{h_1} \times \mathrm{Ran}_{v_1}$.

Special case: $R = [0,1]^2$, $\mathrm{Ran}_A = [0,1]$

Arithmetic mean

Consider the arithmetic mean $A(x,y) = \frac{x+y}{2}$. Then its upper boundaries h_1^A, v_1^A are continuous and given by

$$h(x) := h_1^A(x) = v_1^A(x) = \tfrac{1}{2}(1+x).$$

Then

$$A(x,y) = W(h(x), h(y)) = \max(h(x)+h(y)-1, 0),$$

but also $A(x,y) = C(h(x), h(y))$ with C being the copula

$$C(u,v) = \begin{cases} 0, & \text{if } (u,v) \in [0,1/2]^2\,, \\ \min(u, v-\frac{1}{2}), & \text{if } (u,v) \in [0,1/2[\, \times\,]1/2,1]\,, \\ \min(v, u-\frac{1}{2}), & \text{if } (u,v) \in \,]1/2,1] \times [0,1/2[\,, \\ u+v-1, & \text{if } (u,v) \in [1/2,1]^2\,. \end{cases}$$

Special case: $R = [0,1]^2$, $\mathrm{Ran}_A = [0,1]$

Construction

For every set of margins $\{h_0, v_0, h_1, v_1\}$ and every copula C, the function $A_c\colon [0,1]^2 \to [0,1]$,

$$A_C(x,y) = C(h_1(x), v_1(y))$$

is a 2-increasing binary aggregation operator, whenever $h_1(1) = v_1(1) = 1$ and $h_1(0) = 0$ or $v_1(0) = 0$.

Special case: $R = [0,1]^2$, $\mathrm{Ran}_A = [0,1]$

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However,

$$A_C \in \{f, g, h_1, v_1\}_2$$

with

$$f\colon [0,1] \to [0,1], \quad f(x) := C(h_1(x), v_1(0));$$
$$g\colon [0,1] \to [0,1], \quad g(y) := C(h_1(0), v_1(y)).$$

Special case: $R = [0,1]^2$, $\mathrm{Ran}_A = [0,1]$

Representation

For every 2-increasing binary aggregation operator $A\colon [0,1]^2 \to [0,1]$ with margins $M^A = \{h_0^A, h_1^A, v_0^A, v_1^A\}$, with

$$\lambda_A := V_A([0,1]^2) > 0,$$

there exists a copula C such that

$$A(x,y) = \lambda_A C\left(\frac{h_1^A(x) - h_0^A(x) - h_1^A(0)}{\lambda_A}, \frac{v_1^A(y) - v_0^A(y) - v_1^A(0)}{\lambda_A}\right) + h_0^A(x) + v_0^A(y).$$

for all $x, y \in [0,1]$.

Special case: $R = [0,1]^2$, $\text{Ran}_A = [0,1]$

Construction

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Then, for every copula C, the function $A^C\colon [0,1]^2 \to [0,1]$,

$$A^C(x,y) := \lambda_A C\left(\frac{h_1^A(x) - h_0^A(x) - h_1^A(0)}{\lambda_A}, \frac{v_1^A(y) - v_0^A(y) - v_1^A(0)}{\lambda_A}\right) + h_0^A(x) + v_0^A(y)$$

is a 2-increasing binary aggregation operator.

Special case: $R = [0,1]^2$, $\mathrm{Ran}_A = [0,1]$

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Then, for every copula C, the function $A^C\colon [0,1]^2 \to [0,1]$,

$$A^C(x,y) := \lambda_A C\left(\frac{h_1^A(x)-h_0^A(x)-h_1^A(0)}{\lambda_A}, \frac{v_1^A(y)-v_0^A(y)-v_1^A(0)}{\lambda_A}\right) + h_0^A(x) + v_0^A(y)$$

is a 2-increasing binary aggregation operator.

Moreover,

$$A^C \in \mathcal{M}_2^A.$$

Special case: $R = [0,1]^2$, $\mathrm{Ran}_A = [0,1]$

Bounds

(Durante et al. (2007))

Consider a set of margins $M = \{h_0, h_1, v_0, v_1\}$.
Then, for all $A \in \mathcal{M}_2$

$$A_* \le A \le A^*$$

with

$$A_*(x,y) := \max(h_0(x) + v_0(y), h_1(x) + v_1(y) - 1);$$
$$A^*(x,y) := \min(h_1(x) + v_0(y) - h_1(0), h_0(x) + v_1(y) - h_0(1)).$$

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$$A^*(x,y) := \min(h_1(x) + v_0(y) - h_1(0), h_0(x) + v_1(y) - h_0(1)).$$

Moreover, $A_*, A^* \in \mathcal{M}_2$.

Special case: $R = [0,1]^2$, $\mathrm{Ran}_A = [0,1]$

Bounds

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Then, for all $A \in \mathcal{M}_2$, $A_* \leq A \leq A^*$, with

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The role of λ

- If $A \in \mathcal{M}_2$ and $\lambda_A > 0$, then $A_* = A^W$, $A^* = A^M$.
- If $A \in \mathcal{M}_2$ and $\lambda_A = 0$, then $A_* = A = A^*$.

General case: $R \subset [0,1]^2$

Notation

For every $a, b \in \mathbb{R}$, denote by $\varphi_{a,b}$ the linear transformation

$$\varphi_{a,b} \colon [a, b] \to [0, 1], \quad \varphi_{a,b}(x) := \tfrac{x-a}{b-a}.$$

General case: $R \subset [0,1]^2$

Proposition

Consider a rectangle $R = [a_1, a_2] \times [b_1, b_2]$ with $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$, $b_1 < b_2$.

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If $A\colon [0,1]^2 \to [0,1]$ is a 2-increasing binary aggregation operator, then $A_{\varphi_{a_1,a_2},\varphi_{b_1,b_2},\varphi_{c_1,c_2}} : R \to [0,1]$,

$$A_{\varphi_{a_1,a_2},\varphi_{b_1,b_2},\varphi_{c_1,c_2}}(x,y) := \varphi_{c_1,c_2} \circ A(\varphi_{a_1,a_2}(x), \varphi_{b_1,b_2}(y))$$

is a 2-increasing increasing function and with range $[c_1, c_2]$.

General case: $R \subset [0,1]^2$

Proposition

Consider a rectangle $R = [a_1, a_2] \times [b_1, b_2]$ with $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$, $b_1 < b_2$.

If $A\colon R \longrightarrow [0,1]$ is an increasing 2-increasing function with $\mathrm{Ran}_A = [c_1, c_2]$, then $A_{\varphi^{-1}_{a_1,a_2}, \varphi^{-1}_{b_1,b_2}, \varphi^{-1}_{c_1,c_2}} : [0,1]^2 \longrightarrow [0,1]$,

$$A_{\varphi^{-1}_{a_1,a_2}, \varphi^{-1}_{b_1,b_2}, \varphi^{-1}_{c_1,c_2}}(x,y) := \varphi^{-1}_{c_1,c_2} \circ A(\varphi^{-1}_{a_1,a_2}(x), \varphi^{-1}_{b_1,b_2}(y))$$

is a binary 2-increasing aggregation operator.

General case: $R \subset [0,1]^2$

Representation

Consider

- a rectangle $R = [a_1, a_2] \times [b_1, b_2]$ with $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$, $b_1 < b_2$ and
- a 2-increasing binary function $A\colon R \to [0,1]$ with margins h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2}.

General case: $R \subset [0,1]^2$

Representation

Consider

- a rectangle $R = [a_1, a_2] \times [b_1, b_2]$ with $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$, $b_1 < b_2$ and
- a 2-increasing binary function $A \colon R \to [0,1]$ with margins h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2}.

Then there exists a copula C such that, for all $(x, y) \in R$,

$$A(x,y) = \varphi_{c_1,c_2}^{-1}\Big(C\big(\varphi_{c_1,c_2}(h_{b_2}(x)), \varphi_{c_1,c_2}(v_{a_2}(y))\big)\Big)$$

where $c_1 = A(a_1, b_1)$ and $c_2 = A(a_2, b_2)$.

General case: $R \subset [0,1]^2$

Representation

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- a 2-increasing binary function $A \colon R \to [0,1]$ with margins h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2} such that $\lambda_A > 0$.

General case: $R \subset [0,1]^2$

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Consider

- a rectangle $R = [a_1, a_2] \times [b_1, b_2]$ with $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$, $b_1 < b_2$ and
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Then there exists a copula C such that

$$A = (A^C_{\varphi_{a_1,a_2}, \varphi_{b_1,b_2}, \varphi_{c_1,c_2}})_{\varphi^{-1}_{a_1,a_2}, \varphi^{-1}_{b_1,b_2}, \varphi^{-1}_{c_1,c_2}}$$

where $c_1 = A(a_1, b_1)$ and $c_2 = A(a_2, b_2)$.

General case: $R \subset [0,1]^2$

Construction

Consider

- a rectangle $R = [a_1, a_2] \times [b_1, b_2]$ with $a_1, a_2, b_1, b_2 \in [0, 1]$ such that $a_1 < a_2$, $b_1 < b_2$ and
- a set of margins $M = \{h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2}\}$ such that $\lambda > 0$.

Then, for any copula C,

General case: $R \subset [0,1]^2$

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- a set of margins $M = \{h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2}\}$ such that $\lambda > 0$.

Then, for any copula C, the function $A^C \colon [a_1, a_2] \times [b_1, b_2] \to [0,1]$,

$$A^C(x,y) = \lambda C\left(\frac{V_{14}(x)}{\lambda}, \frac{V_{12}(y)}{\lambda}\right) + h_{b_1}(x) + v_{a_1}(y) - h_{b_1}(a_1)$$

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with

$$V_{14}(x) := h_{b_2}(x) - h_{b_2}(a_1) - h_{b_1}(x) + h_{b_1}(a_1);$$
$$V_{12}(y) := v_{a_2}(y) - v_{a_2}(b_1) - v_{a_1}(y) + v_{a_1}(b_1)$$

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is an increasing 2-increasing function on $[a_1, a_2] \times [b_1, b_2]$.

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with

$$\begin{aligned} V_{14}(x) &:= h_{b_2}(x) - h_{b_2}(a_1) - h_{b_1}(x) + h_{b_1}(a_1); \\ V_{12}(y) &:= v_{a_2}(y) - v_{a_2}(b_1) - v_{a_1}(y) + v_{a_1}(b_1) \end{aligned}$$

is an increasing 2-increasing function on $[a_1, a_2] \times [b_1, b_2]$.

Moreover, $A^C \in \mathcal{M}_2$.

General case: $R \subset [0,1]^2$

Bounds

Consider

- a rectangle $R = [a_1, a_2] \times [b_1, b_2]$ with $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$, $b_1 < b_2$ and
- a set of margins $M = \{h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2}\}$.

Then, for all $A \in \mathcal{M}_2$,

$$A_* \leq A \leq A^*$$

General case: $R \subset [0,1]^2$

Bounds

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- a rectangle $R = [a_1, a_2] \times [b_1, b_2]$ with $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$, $b_1 < b_2$ and
- a set of margins $M = \{h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2}\}$.

Then, for all $A \in \mathcal{M}_2$,

$$A_* \leq A \leq A^*$$

with

$$A_*(x,y) = \max(h_{b_1}(x) + v_{a_1}(y) - h_{b_1}(a_1), h_{b_2}(x) + v_{a_2}(y) - h_{b_2}(a_2)),$$
$$A^*(x,y) = \min(h_{b_2}(x) + v_{a_1}(y) - h_{b_2}(a_1), h_{b_1}(x) + v_{a_2}(y) - h_{b_1}(a_2)).$$

General case: $R \subset [0,1]^2$

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Consider

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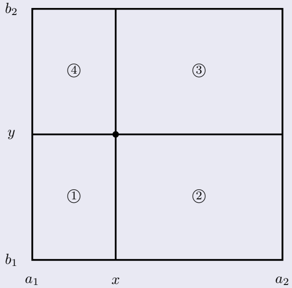
Moreover, $A_*, A^* \in \mathcal{M}_2$.

General case: $R \subset [0,1]^2$

Bounds — geometric interpretation



$$A_*(x,y) = \max(h_{b_1}(x) + v_{a_1}(y) - h_{b_1}(a_1), h_{b_2}(x) + v_{a_2}(y) - h_{b_2}(a_2));$$
$$A^*(x,y) = \min(h_{b_2}(x) + v_{a_1}(y) - h_{b_2}(a_1), h_{b_1}(x) + v_{a_2}(y) - h_{b_1}(a_2)).$$

General case: $R \subset [0,1]^2$

Representation

Consider

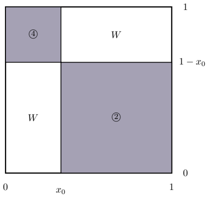
- a rectangle $R = [a_1, a_2] \times [b_1, b_2]$ with $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 < a_2$, $b_1 < b_2$ and
- a 2-increasing binary function $A \colon R \to [0,1]$ with margins h_{b_1}, h_{b_2}, v_{a_1}, v_{a_2} such that $\lambda_A = 0$.

Then $A = A_* = A^*$, i.e., for all $(x,y) \in R$,

$$\begin{aligned} A(x,y) &= h_{b_1}(x) + v_{a_1}(y) - h_{b_1}(a_1) \\ &= h_{b_2}(x) + v_{a_2}(y) - h_{b_2}(a_2) \\ &= h_{b_2}(x) + v_{a_1}(y) - h_{b_2}(a_1) \\ &= h_{b_1}(x) + v_{a_2}(y) - h_{b_1}(a_2). \end{aligned}$$

W-ordinal sums

- $R_2 = [0, x_0] \times [1 - x_0, 1]$:
  Since $\lambda_{A_2} = x_0$, for any copula C_2,
  $$A_2(x, y) = x_0 \, C_2\left(\frac{x}{x_0}, \frac{y + x_0 - 1}{x_0}\right)$$
  for all $(x, y) \in R_2$, is appropriate.
- $R_4 = [x_0, 1] \times [0, 1 - x_0]$:
  Then $\lambda_{A_4} = 1 - x_0$, for any copula C_4,
  $$A_4(x, y) = (1 - x_0) \, C_4\left(\frac{x - x_0}{1 - x_0}, \frac{y}{1 - x_0}\right),$$
  for all $(x, y) \in R_4$, is appropriate.



W-ordinal sums

- $R_1 = [0, x_0] \times [0, 1 - x_0]$:
  Since $\lambda_{A_1} = 0$, therefore

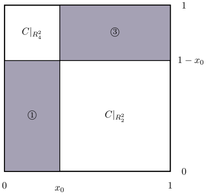
  $$A_1(x, y) = (A_1)_*(x, y) = (A_1)^*(x, y),$$
  $$A_1(x, y) = 0 = W(x, y),$$

  for all $(x, y) \in R_1$.

- $R_3 = [x_0, 1] \times [1 - x_0, 1]$:
  Then $\lambda_{A_3} = 1 - x_0 - (1 - x_0) = 0$, therefore

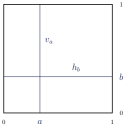
  $$A_3(x, y) = (A_3)_*(x, y) = (A_3)^*(x, y),$$
  $$A_3(x, y) = x + y - 1 = W(x, y),$$

  for all $(x, y) \in R_3$.



Copulas with given horizontal and vertical section

Assume that $C(a,b) = c$ with $0 < c < \min(a,b)$ such that $\lambda_{R_i} > 0$, $i = 1, \dots, 4$.
Choose copulas C_1, C_2, C_3, C_4.



$$A^{C_1}(x,y) = cC_1\left(\frac{h_b(x)}{c}, \frac{v_a(y)}{c}\right),$$

$$A^{C_2}(x,y) = (b-c)C_2\left(\frac{h_b(x)-c}{b-c}, \frac{y}{b-c}\right),$$

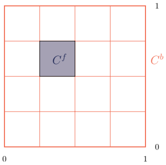
$$A^{C_3}(x,y) = (1+c-a-b)C_3\left(\frac{x+c-h_b(x)-a}{1+c-a-b}, \frac{y+c-v_a(y)-b}{1+c-a-b}\right) + h_b(x) + v_a(y) - c,$$

$$A^{C_4}(x,y) = (a-c)C_4\left(\frac{x-h_b(x)}{a-c}, \frac{v_a(y)-c}{a-c}\right),$$

$$C(x,y) = \begin{cases} A^{C_1}(x,y), & \text{if } (x,y) \in [0,a] \times [0,b], \\ A^{C_2}(x,y), & \text{if } (x,y) \in [a,1] \times [0,b], \\ A^{C_3}(x,y), & \text{if } (x,y) \in [a,1] \times [b,1], \\ A^{C_4}(x,y), & \text{if } (x,y) \in [0,a] \times [b,1]. \end{cases}$$

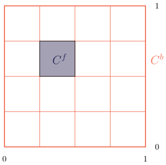
Copulas constructed on an orthogonal grid

- Choice of C_b determines whether $\lambda > 0$ or $\lambda = 0$.



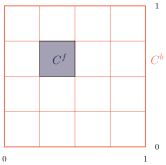
Copulas constructed on an orthogonal grid

- Choice of C_b determines whether $\lambda > 0$ or $\lambda = 0$.
- When $C_b = \Pi$, $\lambda > 0$ for all rectangles,



Copulas constructed on an orthogonal grid

- Choice of C_b determines whether $\lambda > 0$ or $\lambda = 0$.
- When $C_b = \Pi$, $\lambda > 0$ for all rectangles, then for arbitrary foreground copula C_f



$$C(x,y) = \begin{cases} xy - (x-a_1)(y-b_1) \\ \qquad + (a_2-a_1)(b_2-b_1)C_f\left(\frac{x-a_1}{a_2-a_1}, \frac{y-b_1}{b_2-b_1}\right), \\ \qquad\qquad\qquad \text{if } (x,y) \in [a_1, a_2] \times [b_1, b_2]\,, \\ xy, \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Concluding remarks

- Motivation, examples;
- Set of margins and its properties;
- 2-increasing functions with particular domain and range.

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Thank you.