Properties of a new class of bivariate copulas

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1. Introduction

Purpose: Introduce and study a new class of bivariate copulas depending on two univariate functions.

Interest (at a first sight): The new class contains a great variety of copulas:

- ▷ Absolutely continuous copulas, singular copulas and copulas with two components (absolutely continuous and singular).
- ▷ Symmetric and nonsymmetric copulas.
- ▷ It contains families where Kendall's tau and Spearman's rho take any value in $[-1, 1]$.
- ▷ Copulas satisfying different properties of positive and negative dependence.
- ▷ In the literature some examples of the new class has been useful to provide interesting examples and counter-examples.
- ▷ $\ldots$

2. Preliminaries

Let $\mathbb{I} = [0, 1]$. A *copula* is a function $C\colon \mathbb{I}^2 \longrightarrow \mathbb{I}$ which satisfies the *boundary conditions*

$$C(t, 0) = C(0, t) = 0 \quad \text{and} \quad C(t, 1) = C(1, t) = t$$

for all $t \in \mathbb{I}$, and the 2-*increasing property*, i.e.,

$$C(x_2, y_2) - C(x_2, y_1) - C(x_1, y_2) + C(x_1, y_1) \geq 0$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{I}$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

A copula C is *symmetric* if $C(x, y) = C(y, x)$ for every $(x, y) \in \mathbb{I}^2$. The *transposed copula* of a bivariate copula C is given by $C^t(x, y) = C(y, x)$ for every $(x, y) \in \mathbb{I}^2$.

Some notation: Let Π denote the copula of independent random variables, i.e., $\Pi(x, y) = xy$ for every $(x, y) \in \mathbb{I}^2$. The *Fréchet-Hoeffding bounds* for bivariate copulas will be denoted by $M(x, y) = \min(x, y)$ and $W(x, y) = \max(x + y - 1, 0)$ ($(x, y) \in \mathbb{I}^2$).

Let C_1 and C_2 be two copulas. C_2 is said *more concordant* than C_1, written $C_1 \prec C_2$, if $C_1(x, y) \leq C_2(x, y)$ for every $(x, y) \in \mathbb{I}^2$.

The *horizontal sections* of a copula C are the functions f_{y_0} ($y_0 \in \mathbb{I}$) defined on $\mathbb{I}$ by $f_{y_0}(x) = C(x, y_0)$.

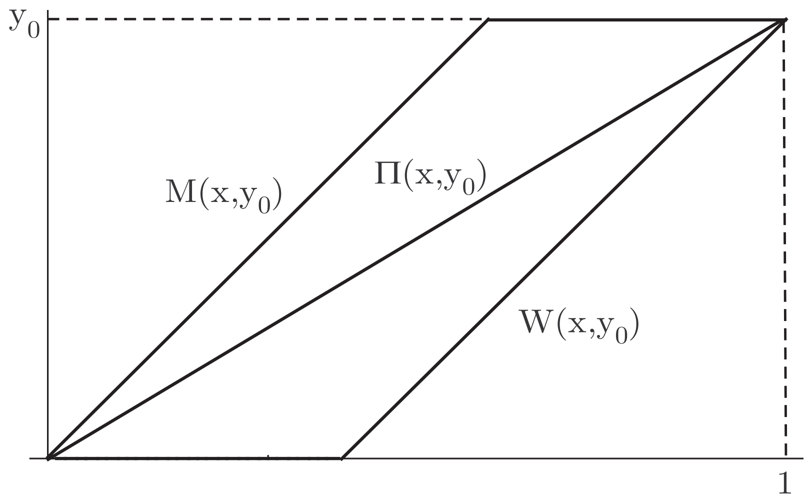

Figure 1: The horizontal sections of W, Π and M

An statistical interpretation of the horizontal sections: if (X, Y) is a random pair with copula C such that their margins are uniform on $\mathbb{I}$, then the horizontal sections of C are proportional to the conditional distribution function: $\text{Pr}[X \leq x | Y \leq y_0]$:

$$C(x, y_0) = y_0 \text{Pr}[X \leq x | Y \leq y_0].$$

In this paper we study copulas whose horizontal sections are a simple type of piecewise linear functions. The simplest examples of these copulas are the copulas Π, M and W. In the literature, we can find different classes of copulas with piecewise linear horizontal or vertical sections, such as

- ▷ the Fréchet and Mardia family of copulas: Fréchet (1958), Mardia (1970)
- ▷ the shuffles of M: Mikusiński et al. (1992)
- ▷ the maximum and minimum copulas (among others) in several classes of copulas with prescribed horizontal or vertical sections, such as copulas with quadratic, cubic and two different types of sinusoidal horizontal or vertical sections: Quesada-Molina and Rodríguez-Lallena (1995), Nelsen et al. (1997) and Rodríguez-Lallena (1996).
- ▷ In general, Corollary 2.6 in Rodríguez-Lallena, J. A. and Úbeda-Flores, M. (2004) suggests that the maximum and minimum copulas in any class of copulas with prescribed horizontal or vertical sections have piecewise linear vertical or horizontal sections, respectively.

(but only some cases in the first two families belongs to the new family)

3. A new class of copulas

We look for copulas whose horizontal sections has the following form:

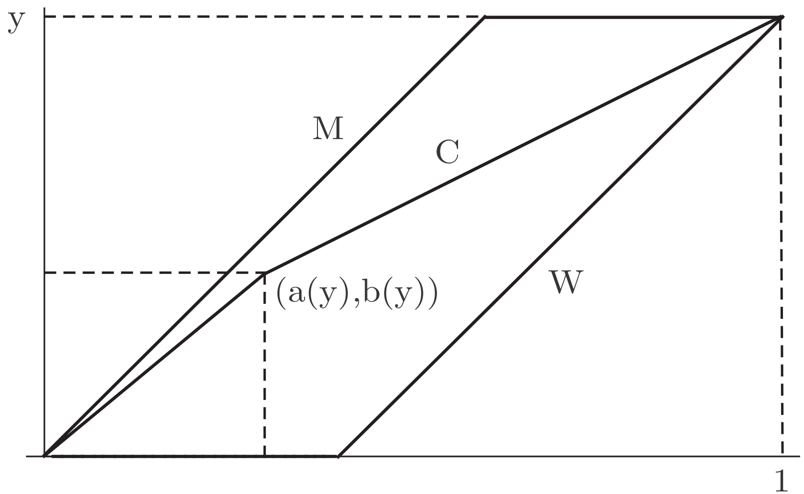

Figure 2: The horizontal sections of W, C and M

The point $(a(y), b(y))$, for every $y \in \mathbb{I}$, generates two real functions a and b on $\mathbb{I}$—which we will suppose continuous—such that $a(y) \in \mathbb{I}$ and

$$\max\left(a(y) + y - 1, 0\right) \leq b(y) \leq \min\left(a(y), y\right) \quad \text{for all } y \in \mathbb{I}.$$

Then, the function $C\colon \mathbb{I}^2 \longrightarrow \mathbb{I}$ whose horizontal sections are the piecewise linear functions shown in the figure is given by

$$C(x,y) = \begin{cases} x\dfrac{b(y)}{a(y)}, & 0 \le x \le a(y), \\ y - (1-x)\dfrac{y-b(y)}{1-a(y)}, & a(y) < x \le 1. \end{cases} \tag{1}$$

For brevity, sometimes we denote $r(y) = \frac{b(y)}{a(y)}$ and $s(y) = \frac{y-b(y)}{1-a(y)}$, whence

$$C(x,y) = \begin{cases} x\,r(y), & 0 \le x \le a(y), \\ y - (1-x)s(y), & a(y) < x \le 1. \end{cases}$$

If the function C is a copula and (X,Y) is a random pair with copula C whose margins are uniform on $\mathbb{I}$, then the conditional distribution function $\Pr[X \le x | Y \le y_0]$ spreads uniformly mass on two subintervals of $\mathbb{I}$: it assigns uniformly a mass of $r(y_0)/y_0$ on the interval $[0, a(y_0)]$ and a mass of $s(y_0)/y_0$ on the interval $[a(y_0), 1]$ when $a(y_0) \in (0,1)$; otherwise, such a conditional distribution is uniform on $\mathbb{I}$.

But, under which conditions is C a copula?

Lemma 1 *Let* $a, b\colon \mathbb{I} \to \mathbb{I}$ *be two continuous functions satisfying the inequality* $\max(a(y) + y - 1, 0) \leq b(y) \leq \min(a(y), y)$ *for all* $y \in \mathbb{I}$. *Then:*

1. $b(y) \in \mathbb{I}$ *for every* $y \in \mathbb{I}$;
2. $y - b(y) \leq 1 - a(y)$ *for every* $y \in \mathbb{I}$;
3. $b(0) = 0$;
4. $b(1) = a(1)$;
5. *If* $a(y) = 0$ *for some* $y \in \mathbb{I}$, *then* $b(y) = 0$;
6. *If* $a(y) = 1$ *for some* $y \in \mathbb{I}$, *then* $b(y) = y$.

Lemma 2 *Let* $a, b\colon \mathbb{I} \to \mathbb{I}$ *be two continuous functions satisfying the inequality* $\max(a(y) + y - 1, 0) \leq b(y) \leq \min(a(y), y)$ *for all* $y \in \mathbb{I}$. *If* C *is the function defined on* $\mathbb{I}^2$ *by* (1), *then we have:*

1. C *is a continuous function from* $\mathbb{I}^2$ *onto* $\mathbb{I}$;
2. C *satisfies the boundary conditions for copulas.*

Therefore, the function C is a copula if and only if C is 2-increasing.

Characterization of copulas of the form (1)

Theorem 3 *Let $a, b\colon \mathbb{I} \to \mathbb{I}$ be two continuous functions satisfying the inequality $\max(a(y)+y-1,0) \leq b(y) \leq \min(a(y),y)$ for all $y \in \mathbb{I}$. Let r and s be the functions respectively defined by*

$$r(y) = \frac{b(y)}{a(y)} \quad \textit{for every } y \in \mathbb{I} \textit{ such that } a(y) \neq 0, \tag{2}$$

$$s(y) = \frac{y-b(y)}{1-a(y)} \quad \textit{for every } y \in \mathbb{I} \textit{ such that } a(y) \neq 1. \tag{3}$$

Then, the function C given by (1) is a copula if and only if the following three conditions hold:

1. *Both r and s are nondecreasing functions on their respective domains.*
2. *$s(y) \leq r(y')$ for every $y, y' \in \mathbb{I}$ such that $y < y'$ and $a(y) < a(y')$.*
3. *$r(y) \leq s(y')$ for every $y, y' \in \mathbb{I}$ such that $y < y'$ and $a(y) > a(y')$.*

The set of copulas of the type characterized in Theorem 3 will be denoted by $\mathcal{P}$. That theorem and every result for $\mathcal{P}$ can be immediately extended to the set $\mathcal{P}^t$ of the transposed copulas to those in the set $\mathcal{P}$.

Sufficient conditions for copulas of the form (1)

Theorem 4 *Let $a, b \colon \mathbb{I} \to \mathbb{I}$ be two continuous functions satisfying the following conditions:*

1. $\max(a(y) + y - 1, 0) \leq b(y) \leq \min(a(y), y)$ *for all* $y \in \mathbb{I}$.
2. *Both* a *and* b *are piecewise derivable, and their derivatives are continuous on their respective domains.*
3. *There exists a finite partition of* $\mathbb{I}$ *such that the derivative* a' *is either positive or negative or zero in the interior of each subinterval of that partition.*
4. $a'(y)(b(y) - ya(y)) \geq 0$ *for every* $y \in \mathbb{I}$ *such that* $a'(y)$ *exists.*
5. $a'(y)\dfrac{b(y)}{a(y)} \leq b'(y)$ *whenever* $a(y) \neq 0$ *and both* $a'(y)$ *and* $b'(y)$ *exist.*
6. $b'(y) \leq 1 + a'(y)\dfrac{y - b(y)}{1 - a(y)}$ *whenever* $a(y) \neq 1$ *and both* $a'(y)$ *and* $b'(y)$ *exist.*

Then, the function C *defined on* $\mathbb{I}^2$ *by* (1) *is a copula.*

4. Properties of the new class of copulas

Absolute continuity

Theorem 5 *Let* $a, b\colon \mathbb{I} \longrightarrow \mathbb{I}$ *be two absolutely continuous functions such that the function* C *given by* (1) *is a copula. Let* $\mathbb{I}_a = \{y \in \mathbb{I} \,:\, 0 < a(y) < 1\}$. *Then,* C *is absolutely continuous if and only if*

$$\int_{\mathbb{I}_a} \frac{a'(y)\,(b(y) - ya(y))}{a(y)\,(1 - a(y))}\, dy = 0.$$

Otherwise, C *has a singular component whose mass is concentrated on the set* $\{(a(y), y) \,:\, y \in \mathbb{I}_a\}$*; and that mass is equal to the integral*

$$\int_{\mathbb{I}_a} \frac{a'(y)\,(b(y) - ya(y))}{a(y)\,(1 - a(y))}\, dy.$$

Measures of association: Spearman's rho and Kendall's tau

Theorem 6 *Let $a, b \colon \mathbb{I} \longrightarrow \mathbb{I}$ be two continuous functions such that the function C given by (1) is a copula. Then,*

$$\rho_C = 6 \int_0^1 (b(y) - ya(y))\, dy.$$

If a and b are derivable almost everywhere in $\mathbb{I}_a$, then,

$$\tau_C = 2 \int_{\mathbb{I}_a} \left[b(y) - yb'(y) + \left(\frac{b^2(y)}{a(y)} + \frac{(y - b(y))^2}{1 - a(y)} \right) a'(y) \right] dy.$$

Concordance ordering

Theorem 7 *For each $i = 1, 2$, let $a_i, b_i \colon \mathbb{I} \longrightarrow \mathbb{I}$ be two continuous functions such that the function C_i given by (1) is a copula; and let r_i and s_i the functions defined by (2) and (3), respectively. Then, $C_1 \prec C_2$ if and only if $r_1(y) \leq r_2(y)$ whenever $a_1(y)a_2(y) \neq 0$ and $s_1(y) \geq s_2(y)$ whenever $(1 - a_1(y))(1 - a_2(y)) \neq 0$.*

Dependence concepts

Next results characterize those continuous random pairs which satisfy certain positive dependence properties (it can be obtained similar results for the corresponding negative dependence concepts). A common hypothesis to those results is the following: a and b are continuous functions such that the function C given by (1) is a copula; and (X, Y) is a continuous random pair with copula C.

Theorem 8 *X and Y are positively quadrant dependent ($\Pi \prec C$) if and only if $b(y) \geq ya(y)$ for all $y \in \mathbb{I}$.*

Theorem 9 *The following statements are equivalent:*

1. *X and Y are positively quadrant dependent ($\mathrm{PQD}(X, Y)$).*
2. *Y is left tail decreasing in X ($\mathrm{LTD}(Y|X)$).*
3. *Y is right tail increasing in X ($\mathrm{RTI}(Y|X)$).*
4. *Y is stochastically increasing in X ($\mathrm{SI}(Y|X)$).*

Theorem 10 *We have the following characterizations of two other concepts of positive dependence:*

1. X *is left tail decreasing in* Y *(*$\mathrm{LTD}(X|Y)$*) if and only if both the functions* $\dfrac{-r(y)}{y}$ *and* $\dfrac{s(y)}{y}$ *are nondecreasing on their respective domains.*

2. X *is right tail increasing in* Y *(*$\mathrm{RTI}(X|Y)$*) if and only if both the functions* $\dfrac{r(y)-1}{1-y}$ *and* $\dfrac{1-s(y)}{1-y}$ *are nondecreasing on their respective domains.*

Theorem 11 *If* $C \neq \Pi$ *and the function* a *is a piecewise monotone function, then we have the following equivalences:*

1. $\mathrm{SI}(X|Y)$ *if, and only if,* $b(y) > ya(y)$ *for all* $y \in (0,1)$*, and both the functions* $-r(y)$ *and* $s(y)$ *are convex.*

2. X *and* Y *are left corner set decreasing (*$\mathrm{LCSD}(X,Y)$*) if and only if* $\mathrm{LTD}(X|Y)$*;*

3. X *and* Y *are right corner set increasing (*$\mathrm{RCSI}(X,Y)$*) if and only if* $\mathrm{RTI}(X|Y)$*.*

Theorem 12 *If C is absolutely continuous and a is a piecewise monotone function, then X and Y are positively likelihood ratio dependent (*$\mathrm{PLR}(X, Y)$*) if and only if a is a constant function and b is concave.*

Symmetry

Theorem 13 *Let (x_0, y_0) be a point in $\mathbb{R}^2$ and let (X, Y) be a pair of continuous random variables whose associated copula C is in $\mathcal{P}$. Suppose that (X, Y) is marginally symmetric about (x_0, y_0). Then:*

1. *(X, Y) is radially symmetric about (x_0, y_0) if and only either $C = \Pi$ (X and Y are independent) or the functions a and b satisfy that $a(y) + a(1 - y) = 1$ and $b(1 - y) - b(y) = 1 - y - a(y)$ for every $y \in \mathbb{I}$.*

2. *(X, Y) is jointly symmetric about (x_0, y_0) if and only if X and Y are independent.*

5. Examples

A family of symmetric absolutely continuous copulas

Example 1 *Let* $\lambda \in (0,1)$ *and* $\kappa \in \mathbb{R}$ *such that* $-\min\left(\frac{\lambda}{1-\lambda}, \frac{1-\lambda}{\lambda}\right) \leq \kappa \leq 1$. *If* $a(y) = \lambda$ *and* $b(y) = (1-\kappa)\lambda y + \kappa \min(y, \lambda)$, *for every* $y \in \mathbb{I}$, *then we have the following family of copulas:*

$$C_{\lambda,\kappa}(x,y) = \begin{cases} xy + \frac{(1-\lambda)\kappa}{\lambda} xy, & \max(x,y) \leq \lambda, \\ xy + \kappa x(1-y), & x \leq \lambda < y, \\ xy + \kappa y(1-x), & y \leq \lambda < x, \\ xy + \frac{\lambda\kappa}{1-\lambda}(1-x)(1-y), & \lambda < \min(x,y). \end{cases}$$

- $\rho_{C_{\lambda,\kappa}} = 3\lambda(1-\lambda)\kappa$ and $\tau_{C_{\lambda,\kappa}} = \frac{2}{3}\rho_{C_{\lambda,\kappa}}$. The maximum and minimum values for ρ are $\rho_{C_{1/2,1}} = 3/4$ and $\rho_{C_{1/2,-1}} = -3/4$.
- For every $\lambda_0 \in (0,1)$, the parametric family of copulas $C_{\lambda_0,\kappa}$ is positively ordered: $C_{\lambda_0,\kappa_1} \prec C_{\lambda_0,\kappa_2}$ whenever $\kappa_1 \leq \kappa_2$. Observe that $C_{\lambda,0} = \Pi$ for all λ.
- If $\kappa > 0$, we have PLR(X,Y); if $\kappa < 0$, the respective concept of negative dependence holds.
- If the pair (X,Y) is marginally symmetric about a point in $\mathbb{R}^2$, we have that (X,Y) is radially symmetric if and only if $\lambda = 1/2$.

A family of singular copulas

Example 2 *Let* $\alpha \in (0,1)$. *If* $a(y) = \max\left(\dfrac{\alpha - y}{\alpha}, \dfrac{y-\alpha}{1-\alpha}\right)$ *and* $b(y) = \max\left(0, \dfrac{y-\alpha}{1-\alpha}\right)$ *for every* $y \in \mathbb{I}$, *then we have the following family of copulas:*

$$C_\alpha(x,y) = \min\left[x, \max\left(0, y - \alpha(1-x)\right)\right], \quad (x,y) \in \mathbb{I}^2.$$

- The support of C_α is the curve $\left\{\left(\max\left(\frac{\alpha-y}{\alpha}, \frac{y-\alpha}{1-\alpha}\right), y\right) : y \in \mathbb{I}\right\}$.
- This family of copulas C_α is negatively ordered, since $C_{\alpha_2} \prec C_{\alpha_1}$ whenever $\alpha_1 \leq \alpha_2$. As a limit, we can define $C_0 = M$ and $C_1 = W$.
- $\rho_{C_\alpha} = \tau_{C_\alpha} = 1 - 2\alpha$; thus, the whole range for ρ and τ is attained by this family of copulas.
- C_α cannot be associated to quadrant dependence random pairs (except for the extreme cases C_0 and C_1).
- C_α cannot be associated to radially symmetric random pairs

A family of copulas which have an absolutely continuous component and a singular component

Example 3 *Let* $\beta \in \mathbb{I}$. *If* $a(y) = y^2$ *and* $b(y) = \beta y^2 + (1-\beta)y^3$ *for every* $y \in \mathbb{I}$, *then we have the following family of copulas:*

$$C_\beta(x,y) = xy + \frac{\beta}{1+y}\left[\min(x,y^2) - xy^2\right], \quad (x,y) \in \mathbb{I}^2.$$

- If $\beta \neq 0$, C_β has a singular component concentrated on the curve $\{(x,\sqrt{x}) : x \in \mathbb{I}\}$ whose C_β-measure is $2\beta(1-\ln 2)$.
- $\rho_{C_\beta} = \beta/2$ and $\tau_{C_\beta} = [(17 - 24\ln 2)\beta^2 + \beta]/3$.
- The family C_β is positively ordered: $C_{\beta_1} \prec C_{\beta_2}$ if and only if $\beta_1 \leq \beta_2$.
- RTI$(X|Y)$ holds, but LTD$(X|Y)$ does not.
- C_β cannot be associated to radially symmetric random pairs

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