Bounds for Functions of Multivariate Risks

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The problem at hand

On some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, let

$$\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$$

be $\mathbb{R}^k$-valued random vectors having given distribution functions

$$F_1, \ldots, F_n : \mathbb{R}^k \to [0, 1].$$

Given a measurable function $\psi : (\mathbb{R}^k)^n \to \mathbb{R}^k$,
the distribution of the vector

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We want to bound the distribution (tail) function of the vector $\psi(\boldsymbol{X})$ from below (above) on

$$\mathfrak{F}(F_1, \ldots, F_n),$$

the set of dfs having $F_1, \ldots, F_n$ as fixed marginals.

The problems at hand become

$$m_\psi(\boldsymbol{s}) := \inf\{\mathbb{P}[\psi(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) < \boldsymbol{s}] : \boldsymbol{X}_i \backsim F_i, 1 \leq i \leq n\}, \boldsymbol{s} \in \mathbb{R}^k,$$
$$M_\psi(\boldsymbol{s}) := \sup\{\mathbb{P}[\psi(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) \geq \boldsymbol{s}] : \boldsymbol{X}_i \backsim F_i, 1 \leq i \leq n\}, \boldsymbol{s} \in \mathbb{R}^k.$$

The Fréchet class $\mathfrak{F}(F_1, \ldots, F_n)$

The set $\mathfrak{F}(F_1, \ldots, F_n)$ is non-empty.

When $\boldsymbol{k} = \mathbf{1}$:

$$m_\psi(s) = 1 - M_\psi(s))$$

and it is easy to describe elements in $\mathfrak{F}(F_1, \ldots, F_n)$ (Sklar's theorem + concept of copula).

When $\boldsymbol{k} > \mathbf{1}$:

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Assumptions on $\psi : (\mathbb{R}^k)^n \to \mathbb{R}^k$

Given k measurable, **increasing** functions $\psi_j : \mathbb{R}^n \to \mathbb{R}, j \in K$, we construct the function $\psi : (\mathbb{R}^k)^n \to \mathbb{R}^k$ as follows:

$$\psi(\boldsymbol{X}) = \psi(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n) = \psi\left(\begin{pmatrix} X_1^1 \\ \vdots \\ X_1^k \end{pmatrix}, \ldots, \begin{pmatrix} X_n^1 \\ \vdots \\ X_n^k \end{pmatrix} \right) = \begin{pmatrix} \psi_1(X_1^1, \ldots, X_n^1) \\ \vdots \\ \psi_k(X_1^k, \ldots, X_n^k) \end{pmatrix}.$$

If $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ is a matrix of risks, ψ can aggregate risks only row-wise, but the aggregation method may differ between rows.

This makes sense if the risks $X_1, \ldots, X_n$ are componentwise homogeneous.

Why working with multivariate marginals

Assuming multivariate marginals allows not only to fix the univariate df of every component of the single multivariate policies, but also the dependence **within** the single policies.

$$
\begin{array}{r}
\text{insurance line } 1 \rightarrow \\
\vdots \\
\text{insurance line } k \rightarrow
\end{array}
\quad
\psi\left(\underbrace{\begin{pmatrix} X_1^1 \\ \vdots \\ X_1^k \end{pmatrix}}_{\text{policy } 1}, \dots, \underbrace{\begin{pmatrix} X_n^1 \\ \vdots \\ X_n^k \end{pmatrix}}_{\text{policy } n}\right) = \begin{pmatrix} X_1^1 + \cdots + X_n^1 \\ \vdots \\ X_1^k + \cdots + X_n^k \end{pmatrix}
$$

Duality theorems

$m_\psi(s)$ and $M_\psi(s)$ are two **linear problems** over a convex feasible space of measures. Therefore, they admit a **dual representation**.

Main Duality Theorem (Ramachandran and Rüschendorf (1995))

$$m_\psi(\mathbf{s}) = \sup\left\{\sum_{i=1}^n \int_{\mathbb{R}^k} f_i dF_i : f_i \in L^1(F_i), i \in N \textit{ with} \right.$$
$$\left. \sum_{i=1}^n f_i(\mathbf{x}_i) \le 1_{(-\infty,\mathbf{s})}(\psi(\mathbf{x})) \textit{ for all } \mathbf{x} \in (\mathbb{R}^k)^n \right\},$$

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Known solutions

$m_\psi(\boldsymbol{s})$ and $M_\psi(\boldsymbol{s})$, as well as their dual counterparts, are very difficult to solve. Solutions are known only in few cases.

- When $n = 2$ and $k = 1$; see Rüschendorf (1982).
- When $n = 2$ and $k > 1$, Li et al. (1996) give $m_+(\boldsymbol{s})$.
- When $n > 2$, the only explicit solution we know is given in Rüschendorf (1982) for the sum of risks uniformly distributed on the unit interval and in our paper for the sum of risks uniformly distributed on the unit hypercube.

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The basic idea in the dual approach

If $\hat{\mathbf{f}} = (\hat{f}_1, \ldots, \hat{f}_n)$ and $\hat{\mathbf{g}} = (\hat{g}_1, \ldots, \hat{g}_n)$ are two set of functions which are admissible for the corresponding dual problems, we have

$$\mathbb{P}[\psi(\boldsymbol{X}) < \boldsymbol{s}] \geq m_\psi(\boldsymbol{s}) \geq \sum_{i=1}^{n} \int_{\mathbb{R}^k} \hat{f}_i dF_i,$$

$$\mathbb{P}[\psi(\boldsymbol{X}) \geq \boldsymbol{s}] \leq M_\psi(\boldsymbol{s}) \leq \sum_{i=1}^{n} \int_{\mathbb{R}^k} \hat{g}_i dF_i.$$

Therefore, even if we do not solve the dual problems, **dual admissible functions provide bounds on the solutions which are conservative from a risk management viewpoint**.

Standard bounds

We call *standard bounds* those bounds obtained by choosing *piecewise-constant* dual choices.

- Standard bounds are those typically obtained from elementary probability; see: Denuit, Genest, and Marceau (1999) for $k = 1$; Li, Scarsini, and Shaked (1996) for $k > 1$ and $\psi = +$.
- The standard bound on m_ψ is sharp only in the case of the sum of two risks. The one for M_ψ fails to be sharp also for $n = 2$.

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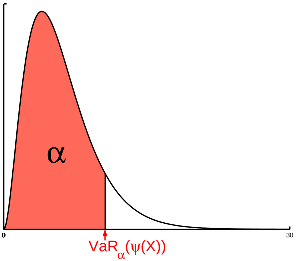
Dual bounds

We call *dual bounds* those bounds obtained by choosing *piecewise-linear* dual choices.

Dual bounds are better than standard bounds when $n > 2$ but actually we can calculate them only for the sum of non-negative risks.

$k = 1$: bounding the Value-at-Risk for an aggregate loss

The Value-at-Risk at probability level α for $\psi(X)$ is the maximum aggregate loss which can occur with probability α, $\alpha \in [0, 1]$.



If G (the df of $\psi(X)$) is strictly increasing,
$\mathrm{VaR}_\alpha(\psi(X))$ is the unique threshold t at which $F(t) = \alpha$, i.e. $F^{-1}(\alpha)$.

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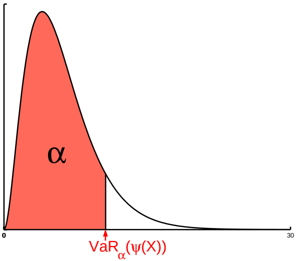

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$\mathrm{VaR}_\alpha(\psi(X))$ is the unique threshold t at which $F(t) = \alpha$, i.e. $F^{-1}(\alpha)$.

Searching for the worst-possible VaR for $\psi(X)$ over $\mathfrak{F}(F_1, \ldots, F_n)$ means looking for

$$m_\psi(s) := \inf\{\mathbb{P}[\psi(X) < s] : X_i \backsim F_i, i = 1, \ldots, n\}, s \in \mathbb{R}.$$

Indeed, according to the definition of VaR, we have

$$\mathrm{VaR}_\alpha(\psi(X)) \leq m_\psi^{-1}(\alpha), \alpha \in [0, 1],$$

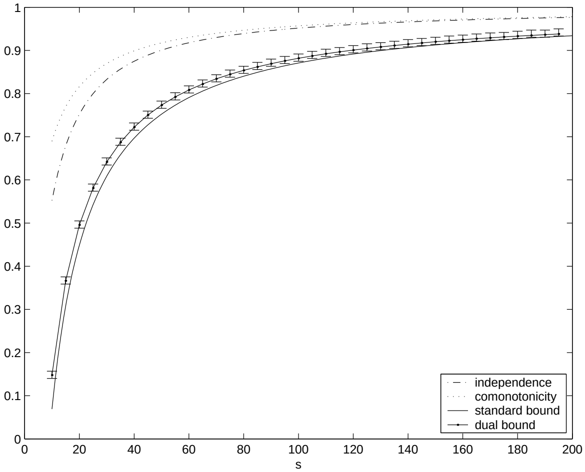

Figure: Range for $\mathbb{P}[X_1 + X_2 + X_3 < s]$ for a Pareto(1.5,1)-portfolio

Bounds on Value-at-Risk

	$\mathrm{VaR}_\alpha(\sum_{i=1}^{10} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{100} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{1000} X_i)$	
α	dual	standard	dual	standard	dual	standard
0.90	0.669	1.485	11.039	149.850	150.162	14998.500
0.95	1.353	2.985	22.227	229.850	301.823	29998.500
0.99	2.985	14.985	111.731	1499.850	1515.111	149998.500
0.999	68.382	149.985	1118.652	14999.850	15164.604	1499998.500

Table: Upper bounds for $\mathrm{VaR}_\alpha(\sum_{i=1}^{n} X_i)$ of three Pareto portfolios of different dimensions. Data in thousands.

For more details on bounding VaR for $k = 1$, homogeneous portfolios of risks, see Embrechts and Puccetti (2006)

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We can obtain the above table also for Moscadelli (2004)'s OR-portfolio.

α	comonotonic value	dual bound	standard bound
0.99	2.8924×10^4	1.4778×10^5	2.6950×10^5
0.995	6.7034×10^4	3.3922×10^5	6.1114×10^5
0.999	4.8347×10^5	2.3807×10^6	4.1685×10^6
0.9999	8.7476×10^6	4.0740×10^7	6.7936×10^7

Table: Range for $\mathrm{VaR}_\alpha\left(\sum_{i=1}^8 X_i\right)$ for the data underlying Moscadelli (2004).

For more details on bounding VaR for $k = 1$, non-homogeneous portfolios of risks, see Embrechts and Puccetti (2006b)

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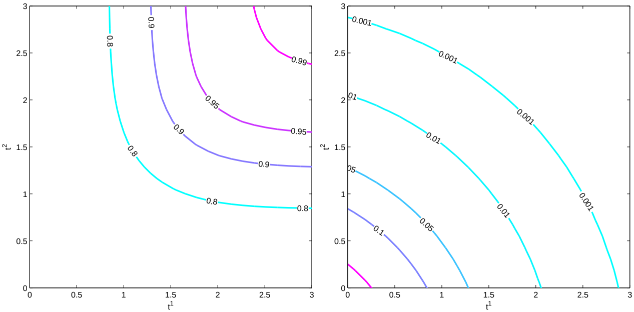
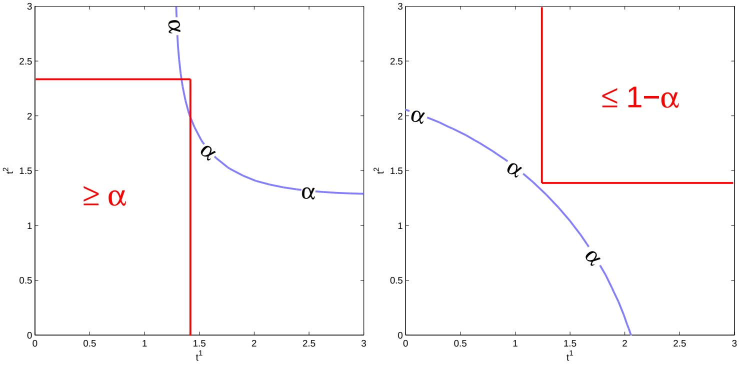
$k > 1$: Multivariate VaR

When $k > 1$, the definition of VaR does not make sense: there are possibly infinitely vectors $\boldsymbol{s} \in \mathbb{R}^k$ at which $G(\boldsymbol{s}) = \alpha$.

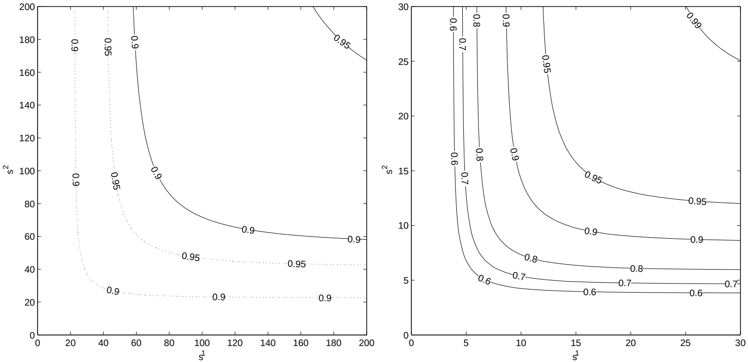


The LO-VaRs for $\psi(\boldsymbol{X})$ are the α-level sets of its df G.
The UO-VaRs are the $(1-\alpha)$-level sets of its tail $\overline{G}$

The LO-VaR$_\alpha$ for m_ψ (left) and the UO-VaR$_\alpha$ for M_ψ (right) provide conservative estimates of the α-VaRs for the aggregate loss $\psi(\boldsymbol{X})$ over $\mathfrak{F}(F_1, \ldots, F_n)$.



$k > 1$: Multivariate Value-at-Risk



Worst-possible LO-VaRs for the sum of two bivariate Pareto ($\theta = 1.2$ for the dotted line) (left) and Log-Normal (right) distributed risks.

Summary

Bounding the df and the tail for an increasing function of dependent random vectors having fixed marginals

↓

general optimal solution is difficult to find when $n > 2$

↓

using the dual formulation of the problem **we can improve the standard bounds obtained from elementary probability**

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dual bounds can be easily computed and can be transformed into bounds on (multivariate) VaR.

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Acknowledgements

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References

Denuit, M., C. Genest, and É. Marceau (1999). Stochastic bounds on sums of dependent risks. *Insurance Math. Econom. 25*(1), 85–104.

Embrechts, P. and G. Puccetti (2006). Bounds for functions of dependent risks. *Finance Stoch. 10*(3), 341–352.

Embrechts, P. and G. Puccetti (2006b). Aggregating risk capital, with an application to operational risk. *Geneva Risk. Insur. Rev. 31*(2), 71–90.

Li, H., M. Scarsini, and M. Shaked (1996). Bounds for the distribution of a multivariate sum. In *Distributions with Fixed Marginals and Related Topics*, Volume 28, pp. 198–212. Hayward, CA: Inst. Math. Statist.

Moscadelli, M. (2004). The modelling of operational risk: experience with the analysis of the data collected by the Basel Committee. Preprint, Banca d'Italia.

Ramachandran, D. and L. Rüschendorf (1995). A general duality theorem for marginal problems. *Probab. Theory Related Fields 101*(3), 311–319.

Rüschendorf, L. (1982). Random variables with maximum sums. *Adv. in Appl. Probab. 14*(3), 623–632.

Scarsini, M. (1989). Copulae of probability measures on product spaces. *J. Multivariate Anal. 31*(2), 201–219.