Quasi-copulas and signed measures

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Main goals:

• Study of (bivariate) quasi-copulas with fractal mass distributions.

• Study of the mass distribution of W^n—the point-wise best-possible lower bound for the set of *n*-quasi-copulas (and *n*-copulas).

• As a consequence, not every multivariate quasi-copula induces a signed measure on $[0, 1]^n$.

1. Copulas and Quasi-copulas

• The importance of *copulas*: Sklar's Theorem [6, 8]. The joint distribution function H of a random vector $(X_1, X_2, \cdots, X_n)$ with respective univariate margins $F_1, F_2, \cdots, F_n$, can be expressed in terms of an n-copula C that is uniquely determined on $\times_{i=1}^{n} \operatorname{Range} F_i$ in the form

$$H(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \cdots, F_n(x_n)), \quad \mathbf{x} \in [-\infty, \infty]^n.$$

• The notion of *quasi-copula* was introduced [1] in order to characterize operations on distribution functions that can, or cannot, be derived from operations on random variables defined on the same probability space.

Theorem 1 [2] *An n-dimensional quasi-copula (or n-quasi-copula) is a function* $Q\colon [0,1]^n \longrightarrow [0,1]$ *that satisfies:*

(Q1) $Q(u_1, \cdots, u_{i-1}, 0, u_{i+1}, \cdots, u_n) = 0$ *and* $Q(1, \cdots, 1, u_i, 1, \cdots, 1) = u_i$ *for all* $\mathbf{u} \in [0,1]^n$ *and for every* $i \in \{1, 2, \cdots, n\}$*;*

(Q2) Q *is nondecreasing in each variable; and*

(Q3) the Lipschitz condition: $|Q(\mathbf{u}) - Q(\mathbf{v})| \leq \sum_{i=1}^{n} |u_i - v_i| \;\; \forall\, \mathbf{u}, \mathbf{v} \in [0,1]^n.$

• Every n-copula is an n-quasi-copula; and when Q is an n-quasi-copula but not an n-copula, it is said that Q is a *proper* n-quasi-copula.

• Every n-quasi-copula Q satisfies that, for every $\mathbf{u} \in [0,1]^n$,

$$W^n(\mathbf{u}) = \max\Big(0, \sum_{i=1}^{n} u_i - n + 1\Big) \leq Q(\mathbf{u}) \leq \min(\mathbf{u}) = M^n(\mathbf{u}).$$

• M^n is an n-copula for all $n \geq 2$, W^2 is a 2-copula, and W^n ($n \geq 3$) is a proper n-quasi-copula.

• For an n-quasi-copula Q and an *n-box* $B = \times_{i=1}^{n}[a_i, b_i]$ in $[0,1]^n$, the *Q-volume* of B is defined similarly than for n-copulas, i.e.,

$$V_Q(B) = \sum \operatorname{sgn}(\mathbf{c}) \cdot Q(\mathbf{c}).$$

We refer to V_Q as the *mass distribution* of Q, and $V_Q(B)$ the *mass accumulated* by Q on B.

The importance of quasi-copulas:

• Quasi-copulas (and copulas) are a special type of *aggregation operators*.

• The set of quasi-copulas is a complete lattice.

2. Signed measures

• Every n-copula C induces a positive measure μ_C defined on the Lebesgue σ-algebra for $[0, 1]^n$.

• Let λ_n denote the Lebesgue measure in $\mathbb{R}^n$. The measure μ_C is *stochastic*, i.e., for every Lebesgue measurable (L-m) set A in $[0, 1]$

$$\mu_C([0, 1]^{i-1} \times A \times [0, 1]^{n-i}) = \lambda_1(A).$$

• μ_C is characterized by the fact that $\mu_C(B) = V_C(B)$ for every n-box B.

Definition 1 [5] *A signed measure μ on a measurable space $(S, \mathcal{A})$ is an extended real valued, countably additive set function on the σ-algebra $\mathcal{A}$ such that $\mu(\emptyset) = 0$, and such that μ assumes at most one of the values $+\infty$ and $-\infty$.*

• Many proper n-quasi-copulas Q induce signed measures μ_Q on $[0, 1]^n$ in the sense that $\mu_Q(B) = V_Q(B)$ for every n-box B. Such measures must satisfy:

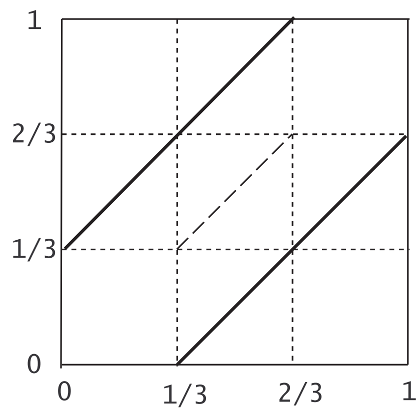
(a) for every L-m set A in $[0, 1]$, $\mu_Q([0, 1]^{i-1} \times A \times [0, 1]^{n-i}) = \lambda_1(A)$,

(b) for every $\mathbf{u} \in [0, 1]^n$, if $u_i \leq v_i \leq 1$ for some $i = 1, 2, \cdots, n$, then

$$0 \leq \mu_Q([0, u_1] \times \cdots \times [0, u_{i-1}] \times [u_i, v_i] \times [0, u_{i+1}] \times \cdots \times [0, u_n]) \leq v_i - u_i.$$

Example 1 *Let s_1, s_2 and s_3 be three segments in $[0,1]^2$, respectively defined by $f_1(x) = x + 1/3$ if $x \in [0, 2/3]$, $f_2(x) = x$ if $x \in [1/3, 2/3]$, and $f_3(x) = x - 1/3$ if $x \in [1/3, 1]$. We spread a mass of $2/3$ uniformly on each of s_1 and s_3, and a mass of $-1/3$ uniformly on s_2. Let $(u,v) \in [0,1]^2$. If we define $Q(u,v)$ as the net mass in the 2-box $[0,u] \times [0,v]$, then Q is a 2-quasi-copula. To be exact:*

$$Q(u,v) = \min(u, v, \max(0, u+v-1, u-1/3, v-1/3)) \quad \forall\,(u,v) \in [0,1]^2.$$



Thus, there exists a signed measure μ_Q such that $\mu_Q(B) = V_Q(B)$ for every 2-box B in $[0,1]^2$ (the difference between the positive measure μ_Q^+ obtained by spreading a mass of $2/3$ uniformly on each of s_1 and s_3, and the positive measure μ_Q^- obtained by spreading a mass of $1/3$ uniformly on s_2).

3. Bivariate quasi-copulas with fractal mass distributions

In [4], the authors construct families of 2-copulas whose supports are fractals [3] by using an iterated function system. We now extend some of the results in [4] to the case of quasi-copulas.

Definition 2 *A quasi-transformation matrix is a matrix* $T = (t_{ij})$, $i = 1, 2, \cdots, m$ *and* $j = 1, 2, \cdots, n$, *with the column index first and the rows ordered from bottom to top, with entries between* $-1/3$ *and* 1, *for which the sum of the entries is* 1, *no row or column has every entry zero, the negative entries are not in the first and the last row or column, e.g.* t_{1j}, t_{mj}, t_{i1}, *and* t_{in} *for all* $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, *and the sum of the entries in any submatrix of the form* (t_{ij}) *for* $i = a, a+1, \cdots, b$, $j = c, c+1, \cdots, d$ *is nonnegative when* $a = 1$ *or* $b = 1$ *or* $c = m$ *or* $d = n$.

Example 2 *Let* T *be the quasi-transformation matrix given by*

$$T = \begin{pmatrix} 0 & 1/3 & 0 \\ 1/3 & -1/3 & 1/3 \\ 0 & 1/3 & 0 \end{pmatrix}. \tag{1}$$

Let $R_{ij} = [p_{i-1}, p_i] \times [q_{j-1}, q_j]$ be a 2-box in $[0,1]^2$ such that p_i (respectively, q_j) denotes the sum of the entries in the first i columns (respectively, j rows) of T. Then, for any 2-quasi-copula Q, let $T(Q)$ be the 2-quasi-copula which, for each (i,j), spreads mass t_{ij} on R_{ij} in the same (but re-scaled) way in which Q spreads mass on $[0,1]^2$, i.e.,

$$T(Q)(u,v) = \sum_{i'<i, j'<j} t_{i'j'} + \frac{u - p_{i-1}}{p_i - p_{i-1}} \cdot \sum_{j'<j} t_{ij'} + \frac{v - q_{j-1}}{q_j - q_{j-1}} \cdot \sum_{i'<i} t_{i'j}$$
$$+ t_{ij} \cdot Q\Big(\frac{u - p_{i-1}}{p_i - p_{i-1}}, \frac{v - q_{j-1}}{q_j - q_{j-1}}\Big),$$

where empty sums are defined to be zero.

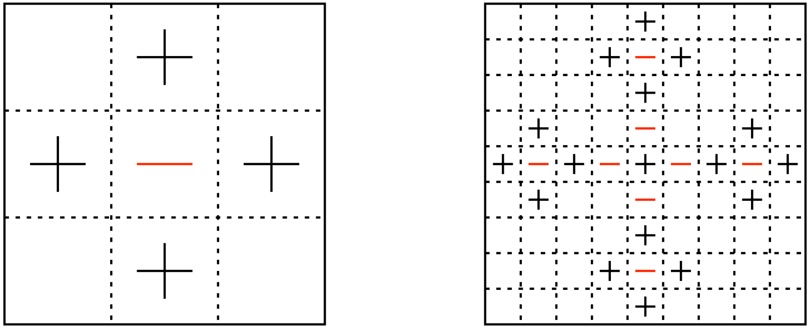
Definition 3 *Let T be a quasi-transformation matrix. For any 2-quasi-copula Q, we define $T^m(Q) = T(T^{m-1}(Q))$, $m = 1, 2, \cdots$, where $T^0(Q) = Q$.*

Theorem 2 *For each quasi-transformation matrix $T \neq (1)$, there is a unique 2-quasi-copula Q_T for which $T(Q_T) = Q_T$. Moreover, Q_T is the limit of the sequence $\{Q, T(Q), T^2(Q), \cdots\}$ for any 2-quasi-copula Q.*

Let T be the quasi-transformation matrix given by (1), i.e.,

$$T = \begin{pmatrix} 0 & 1/3 & 0 \\ 1/3 & -1/3 & 1/3 \\ 0 & 1/3 & 0 \end{pmatrix},$$

and let Π^2 be the copula of independent random variables, i.e., $\Pi^2(u,v) = uv$ for all $(u,v) \in [0,1]^2$. The mass distributions of $T(\Pi^2)$ and $T^2(\Pi^2)$:



For each iteration $m \geq 1$:

- $R_m^+ = 3^m + \sum_{i=0}^{m-1} 3^i \cdot 5^{m-i-1}$ (number of 2-boxes with (+) mass).
- $R_m^- = 5^m - 3^m - \sum_{i=0}^{m-1} 3^i \cdot 5^{m-i-1}$ (number of 2-boxes with (−) mass).
- Total (+) mass $T^m(\Pi^2)$: $1 + (1/5) \sum_{i=0}^{m-1} (5/3)^{m-i}$.
- Total (−) mass $T^m(\Pi^2)$: $-(1/5) \sum_{i=0}^{m-1} (5/3)^{m-i}$.
- $V_{T^m(\Pi^2)}([1/3, 2/3]^2) = -1/3 \Rightarrow T^m(\Pi^2)$ is a proper 2-quasi-copula.

Let $\Pi_T^2 = \lim_{m \to \infty} T^m(\Pi^2)$.

- $V_{\Pi_T^2}([1/3, 2/3]^2) = -1/3 \Rightarrow \Pi_T^2$ is a proper 2-quasi-copula.
- Total (+) mass of Π_T^2: $+\infty$.
- Total (−) mass of Π_T^2: $-\infty$.
- $\lim_{m \to \infty} \lambda_2(R_m^+) = \lim_{m \to \infty} \lambda_2(R_m^-) = 0$.

4. The mass distribution of W^n

For any integer $k \geq 2$, let $T_k = \{1, 2, \cdots, k\}$. We divide $[0,1]^n$ in the k^n following n-boxes:

$$R(i_1, i_2, \cdots, i_n) = \prod_{j=1}^{n} [\frac{i_j - 1}{k}, \frac{i_j}{k}], \ (i_1, i_2, \cdots, i_n) \in T_k^n.$$

Theorem 3 *Let n and k be two integers such that $k \geq n-1 \geq 1$. Let $m = \sum_{j=1}^{n} i_j - k(n-1)$. Then*

$$V_{W^n}(R(i_1, i_2, \cdots, i_n)) = \begin{cases} \dfrac{(-1)^{m-1}}{k} \cdot \dbinom{n-2}{m-1}, & 1 \leq m \leq n-1, \\ 0, & \textit{otherwise.} \end{cases}$$

Moreover, for each m, with $1 \leq m \leq n-1$, the number of n-boxes $R(i_1, i_2, \cdots, i_n)$ satisfying that $V_{W^n}(R(i_1, i_2, \cdots, i_n)) = \frac{(-1)^{m-1}}{k}\binom{n-2}{m-1}$ is $\binom{n-1+k-m}{n-1}$.

Theorem 4 *Let n be an integer such that $n \geq 3$, and let M be any positive real number. Then, there exists a finite set of n-boxes $\{J_1, J_2, \cdots, J_p\}$ in $[0,1]^n$ whose interiors are pairwise disjoint and such that*

(a) $\sum_{i=1}^{p} V_{W^n}(J_i) > M$, *and*

(b) $\sum_{i=1}^{p} \lambda_n(J_i) < 1/M$;

similarly, we can also find a finite set of n-boxes $\{J'_1, J'_2, \cdots, J'_q\}$ in $[0,1]^n$ with pairwise disjoint interiors such that

(c) $\sum_{i=1}^{q} V_{W^n}(J'_i) < -M$, *and*

(d) $\sum_{i=1}^{q} \lambda_n(J'_i) < 1/M$.

Corollary 5 *For every $n \geq 3$, W^n does not induce a signed measure on $[0,1]^n$.*

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