Multivariate Archimedean Copulas

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Sklar's theorem for survival functions

Let $\bar{H}$ be a d-dimensional joint survival function with marginals $\bar{F}_1, \ldots, \bar{F}_d$. Then there always exists a survival copula $\bar{C}$ so that, for any $(x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$\bar{H}(x_1, \ldots, x_d) = \bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d)).$$

If the marginals are continuous then $\bar{C}$ is unique.

And conversely, if $\bar{C}$ is a copula and $\bar{F}_1, \ldots, \bar{F}_d$ are (arbitrary) univariate marginal survival functions, then

$$\bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_d(x_d)) \equiv \bar{H}(x_1, \ldots, x_d)$$

defines a d-dimensional survival function with marginals $\bar{F}_1, \ldots, \bar{F}_d$.

Archimedean copulas

A copula is called Archimedean if it can be written in the form

$$C(u_1, \ldots, u_d) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))$$

for some generator function ψ and its generalized inverse ψ^{-1}.

The generator ψ satisfies

- $\psi : [0, \infty) \to [0, 1]$ with $\psi(0) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$
- ψ is continuous
- ψ is strictly decreasing on $[0, \psi^{-1}(0)]$
- ψ^{-1} is given by $\psi^{-1}(x) = \inf\{u : \psi(u) \leq x\}$

Basic questions

- Is $\psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))$ indeed a copula?
- What is the interpretation of $\psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))$?
- What are the properties of $\psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))$?
- How to sample from $\psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))$?
- How to obtain interesting parametric families, especially when $d \geq 3$?

What conditions on ψ have to be ensured?

Ling (1965)

ψ generates a bivariate copula if and only if ψ is convex.

Kimberling (1974)

ψ generates an Archimedean copula in any dimension if and only if ψ is completely monotone, i.e. $\psi \in C^{\infty}(0, \infty)$ and $(-1)^k \psi^{(k)}(x) \geq 0$ for $k = 1, \dots$.

Nelsen, Genest & Rivest, Müller & Scarsini ...

A generator ψ induces an Archimedean copula in dimension d if $\psi \in C^d(0, \infty)$ and $(-1)^k \psi^{(k)}(x) \geq 0$ for any $k = 1, \dots, d$.

A counterexample

Consider the generator

$$\psi_d^{\mathbf{L}}(x) = \max\left((1-x)^{d-1}, 0\right), \quad x \in (0,\infty).$$

- The d-order derivative of $\psi_d^{\mathbf{L}}$ does not exist for $x = 1$.
- Nonetheless, $\psi_d^{\mathbf{L}}$ can generate a copula in dimension d.

Necessary and sufficient conditions on ψ

ψ generates an Archimedean copula in dimension d if and only if ψ is d-monotone, that is:

- ✓ ψ has continuous derivatives on $(0, \infty)$ up to the order $d-2$.
- ✓ $(-1)^k \psi^{(k)}(x) \geq 0$ for any $k = 1, \ldots, d-2$.
- ✓ $(-1)^{d-2} \psi^{(d-2)}$ is non-negative, non-increasing and convex on $(0, \infty)$.

The Clayton family

Consider the generator

$$\psi_\theta(x) = \max\left((1+\theta x)^{-\frac{1}{\theta}}, 0\right), \quad x \in (0,\infty).$$

- ψ_θ is completely monotone for $\theta \geq 0$.
- ψ_θ is d-monotone for $\theta \geq -\frac{1}{d-1}$.
- ψ_θ is not d-monotone for $\theta < -\frac{1}{d-1}$.
- ψ_θ can generate an Archimedean copula in dimension d if and only if $\theta \geq -\frac{1}{d-1}$.

A detour to real analysis

Williamson d-transform of a non-negative r.v. $R \sim F_R$ is given by

$$\mathfrak{W}_d F_R(x) = \int_{(x,\infty)} \left(1 - \frac{x}{t}\right)^{d-1} dF_R(t), \quad x \in [0,\infty).$$

R. E. Williamson (1956) says:

ψ is a d-monotone (Archimedean) generator if and only if

$$\psi(x) = \mathfrak{W}_d F_R(x)$$

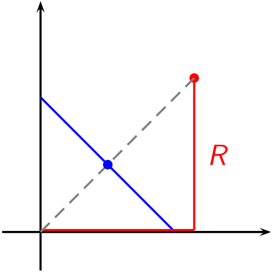
for a non-negative r.v. $R \sim F_R$ with no atom at zero.

F_R is uniquely specified by its Williamson d-transform $\mathfrak{W}_d F_R$.

Simplex distributions ➠ ➠ Archimedean copulas

- ✓ Take a non-negative random variable R with no atom at zero.
- ✓ Take a random vector $\mathbf{S}_d$ independent of R and uniform on

$$\mathcal{S}_d = \left\{\mathbf{x} \in \mathbb{R}_+^d : |x_1| + \cdots + |x_d| = 1\right\}.$$



The survival copula of

$$\mathbf{X} \stackrel{\mathrm{d}}{=} R\mathbf{S}_d$$

is Archimedean with generator

$$\psi(x) = \mathfrak{W}_d F_R(x), \quad x \in [0, \infty).$$

Archimedean copulas ➠➠ Simplex distributions

If $C(\mathbf{u}) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d))$ and $\mathbf{U} \sim C$, then

$$\mathbf{X} \stackrel{\mathrm{d}}{=} (\psi^{-1}(U_1), \ldots, \psi^{-1}(U_d))$$

follows a simplex distribution with no atom at zero.

Furthermore, the distribution function of the radial part is

$$F_R(x) = 1 - \sum_{k=0}^{d-2} \frac{(-1)^k x^k \psi^{(k)}(x)}{k!} - \frac{(-1)^{d-1} x^{d-1} \psi_+^{(d-1)}(x)}{(d-1)!}.$$

An universal sampling recipe ☺

1. Generate R from

$$F_R(x) = 1 - \sum_{k=0}^{d-2} \frac{(-1)^k x^k \psi^{(k)}(x)}{k!} - \frac{(-1)^{d-1} x^{d-1} \psi_+^{(d-1)}(x)}{(d-1)!}.$$

2. Generate independently $\mathbf{S}_d$ using

$$\mathbf{S}_d \stackrel{\mathrm{d}}{=} \left(\frac{Y_1}{Y_1 + \cdots + Y_d}, \ldots, \frac{Y_d}{Y_1 + \cdots + Y_d} \right)$$

where $Y_1, \ldots, Y_d$ are iid with $Y_i \sim \mathrm{Exp}(1)$.

3. Return

$$\left(\psi\left(R \frac{Y_1}{Y_1 + \cdots + Y_d} \right), \ldots, \psi\left(R \frac{Y_d}{Y_1 + \cdots + Y_d} \right) \right).$$

A simple goodness-of-fit procedure ☺☺

Ingredients

Let C be a d-dimensional Archimedean copula C with generator ψ. Then

$$(U_1, \ldots, U_d) \sim C \quad \Rightarrow \quad Y = \psi^{-1}(U_1) + \cdots + \psi^{-1}(U_d) \stackrel{\mathrm{d}}{=} R,$$

$$(U_1, \ldots, U_d) \sim C \quad \Rightarrow \quad \mathbf{V} = \left(\frac{\psi^{-1}(U_1)}{Y}, \ldots, \frac{\psi^{-1}(U_d)}{Y} \right) \stackrel{\mathrm{d}}{=} \mathbf{S}_d.$$

Numerical tests

- Test whether Y and V_j are independent, $j = 1, \ldots, d$.
- Test whether $(1 - V_j)^{d-1}$, $j = 1, \ldots, d$ are standard uniform.

Construction of new families ☺☺☺

- Choose a parametric class of non-negative distributions with no atoms at zero

$$\mathcal{R}_\Theta = \{F_\theta : \theta \in \Theta\}.$$

- Consider

$$\mathcal{C}_\Theta = \{C_\theta : \theta \in \Theta\}$$

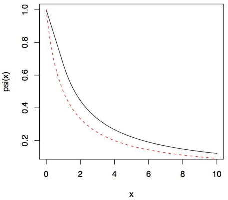
where C_θ is a d-dimensional Archimedean copula generated by

$$\psi_\theta(x) = \mathfrak{W}_d F_\theta(x) = \int_{(x,\infty)} \left(1 - \frac{x}{t}\right)^{d-1} dF_\theta(t).$$

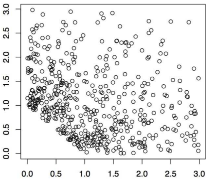
Example: Cut-off

Consider $R \sim F_\theta$ corresponding to the Clayton copula and take

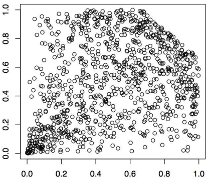
$$R^* \sim F_{\theta,a} \quad \text{where} \quad F_{\theta,a}(x) = \mathbb{P}(R \leq x | R > a)$$



$\psi_{\theta,a}$ and ψ_θ



simplex distribution

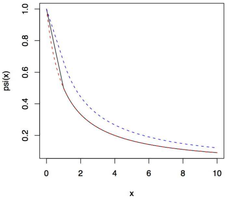


survival copula

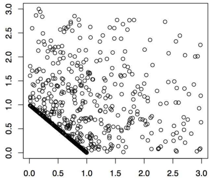
Example: Truncation

Consider $R \sim F_\theta$ corresponding to the Clayton copula and take

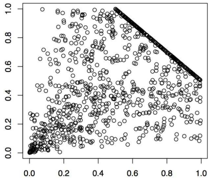
$$\tilde{R} \stackrel{d}{=} \mathbf{1}\{R \le t\}t + \mathbf{1}\{R > t\}R$$



$\psi_{\theta,t}$, $\psi_{\theta,a}$ and ψ_θ



simplex distribution

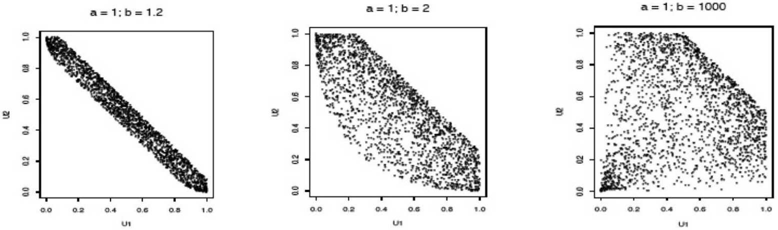


survival copula

Example: Cut-off from both sides

Consider a radial part R with a density

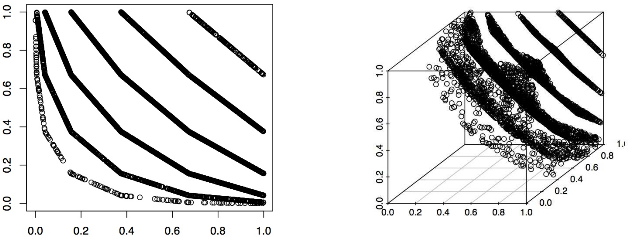
$$f_{a,b}(x) = \frac{ab}{b-a} x^{-2}, \qquad a \leq x \leq b, \quad 0 < a < b.$$



Example: The zebra family

Consider a discrete radial part $R \sim F_{n,p}$, $n \in \mathbb{N}$, $p \in [0,1]$:

$$\mathbb{P}(R = k) = \binom{n}{(k-1)} p^{k-1}(1-p)^{n-k+1}, \qquad k = 1, \ldots, n+1$$

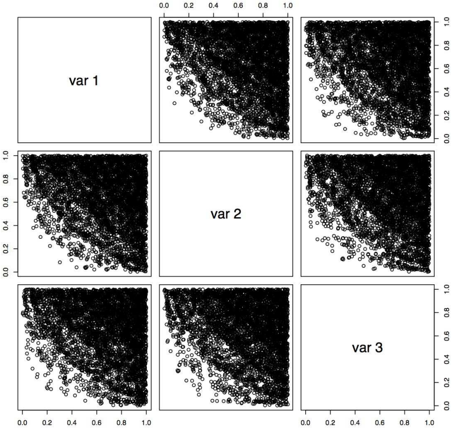


When does an Archimedean copula have a density?

Consider an d-dimensional Archimedean copula C with generator ψ and let R denote the radial part of the corresponding simplex distribution. Then

- C has a density if and only if R has a density.
- C has a density if and only if $\psi^{(d-1)}$ is abs. cont. on $(0, \infty)$.
- If ψ generates an Archimedean copula in dimension at least $d + 1$ then C has a density.

In particular, all lower dimensional marginals of an Archimedean copula have densities, even if R is purely discrete!



Level sets of Archimedean copulas

Level sets of a copula are

$$L(s) = \left\{ \mathbf{u} \in [0,1]^d : C(\mathbf{u}) = s \right\}, \qquad s \in [0,1].$$

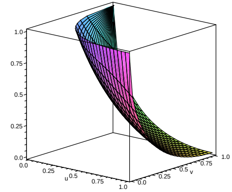
For a d-dimensional Archimedean copula:

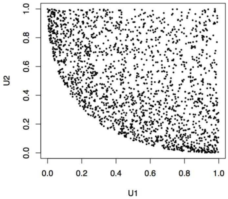
- $$\mathbb{P}^C(L(s)) = \frac{(-1)^{d-1}(\psi^{-1}(s))^{d-1}}{(d-1)!} \left\{ \psi_-^{(d-1)}(\psi^{-1}(s)) - \psi_+^{(d-1)}(\psi^{-1}(0)) \right\}$$
- $$\mathbb{P}^C(L(0)) = \begin{cases} \frac{(-1)^{d-1}(\psi^{-1}(0))^{d-1}\psi_-^{(d-1)}(\psi^{-1}(0))}{(d-1)!} & \text{if } \psi^{-1}(0) < \infty \\ 0 & \text{otherwise} \end{cases}$$

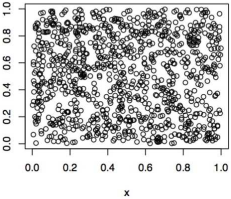
Archimedean copulas are bonded below by a copula ☺

A d-dimensional Archimedean copula with generator ψ satisfies

$$\psi_d^{\mathbf{L}}\left((\psi_d^{\mathbf{L}})^{-1}(u_1)+\cdots+(\psi_d^{\mathbf{L}})^{-1}(u_d)\right)\leq\psi(\psi^{-1}(u_1)+\cdots+\psi^{-1}(u_d)).$$



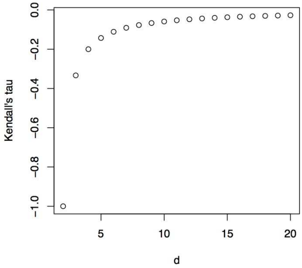




Lower bound on Kendall's tau

For a bivariate margin of a d-dimensional Archimedean copula,

$$\tau = 4\,\mathrm{E}\big(\psi(R)\big) - 1 \quad \text{and} \quad -\frac{1}{2d-3} \leq \tau$$

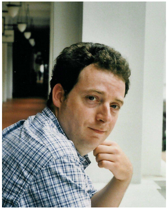


If you want to know more ...

Consult

McNeil, A.J. and Neslehova, J. (2007) *Multivariate Archimedean Copulas, d-monotone Functions and ℓ_1-norm Symmetric Distributions*, FIM Preprint, ETH Zurich.

or



Examples

Consider again

$$\psi_d^{\mathbf{L}}(x) = \max\left((1-x)^{d-1}, 0\right).$$

The radial part of the corresponding simplex distribution satisfies $R = 1$ a.s. and hence $\psi_d^{\mathbf{L}}$ generates the survival copula of $\mathbf{S}_d$.

Distribution function of the radial part corresponding to the bivariate Clayton copula is

$$F_R(x) = 1 - (1+\theta x)^{-\frac{1}{\theta}}\left(1 + \frac{x}{1+\theta x}\right).$$

Simulation procedure

$$\psi(x) = \max\left((1 - x^{1/\theta}), 0\right), \quad \theta \geq 1$$