Sibuya's Dependence Function: a Copula Alternative

Nikolai Kolev[1], **Marcelo Goncalves**[1] **and Boyan Dimitrov**[2]
[1]Department of Statistics, University of Sao Paulo, Brazil
[2]Science and Math. Department, Kettering University, USA.
nkolev@ime.usp.br, mgonca@ime.usp.br
and *bdimitro@kettering.edu*

Outline

1. Introduction (copulas)
2. Suggestion (Sibuya's function)
3. Properties
4. Advantages and disadvantages

Introduction

Let $H(x,y)$ be the joint distribution function of continuous random variables X and Y with marginal distributions $F(x)$ and $G(y)$ defined in $\Re = (-\infty, \infty)$.

Denote by $F^{-1}(u) = inf\{x \in \Re, F(x) \geq u\},\ u \in (0,1)$ and $G^{-1}(v) = inf\{y \in \Re, G(y) \geq v\},\ v \in (0,1)$ the corresponding inverse functions.

Sklar's theorem, (e.g., Sklar (1959)), states that there exists a unique copula function

$$C(u, v) = H(F^{-1}(u), G^{-1}(v)), \quad u, v \in (0, 1)$$

that connects $H(x, y)$ to $F(x)$ and $G(y)$ via

$$H(x, y) = C(F(x), G(y)), \quad x, y \in (-\infty, \infty).$$

Hence, the information in the joint distribution $H(x, y)$ is decomposed into those of marginal distributions and that of copula function $C(u, v)$ which captures the dependence structure between X and Y.

On the other hand, for any copula function $C(u, v)$ and any univariate continuous distribution functions $F(x)$ and $G(y)$, the function $C(F(x), G(y))$ is a bivariate distribution function $H(x, y)$ as given by

$$H(x, y) = C(F(x), G(y)), \quad x, y \in (-\infty, \infty).$$

Consequently, copulas allow one to model the marginal distributions and the dependence structure of multivariate random variable separately.

The copula function is therefore a class of multivariate distributions being functionally independent of its marginals.

Example 1

Consider the symmetric Gumbel bivariate logistic distribution

$$H_1(x,y) = [1 + \exp(-x) + \exp(-y)]^{-1},$$

for all $x, y \in \Re$ with marginal distributions

$$F_1(x) = [1 + \exp(-x)]^{-1} \quad \text{and} \quad G_1(y) = [1 + \exp(-y)]^{-1},$$

and the asymmetric joint distribution

$$H_2(x,y) = \begin{cases} \frac{(x+1)[\exp(y)-1]}{x+2\exp(y)-1}, & \text{if } (x,y) \in [-1,1] \times [0,\infty); \\ 1 - \exp(-y), & \text{if } (x,y) \in (1,\infty) \times [0,\infty); \\ 0, & \text{elsewhere,} \end{cases}$$

with marginals

$$F_2(x) = \frac{x+1}{2},\ x \in [-1,1] \quad \text{and} \quad G_2(y) = 1 - \exp(-y),\ y \geq 0.$$

It is known (e.g. Nelsen, (2006)), that the copula function corresponding to $H_1(x,y)$ and $H_2(x,y)$ is the same, i.e.

$$C(u,v) = \frac{uv}{u+v-uv}.$$

Theoretically, this fact is not surprising, but it is confusing and difficult to be explained for the practitioners.

Note, that $H_1(x,y)$ and $H_2(x,y)$ have completely different support, marginal and symmetric behavior.

The copula function $C(u,v)$ is independent of marginals, and thus, copula is only a class of dependence functions.

The geometrical behavior of the marginal densities (being increasing, decreasing, constant, unimodal functions, functions with a minimum, etc.), have influence on the two-dimensional dependent structure, as demonstrated by Fernandez and Kolev (2007).

The conclusions of this study just indicate that one should search for new classes of dependent functions, in which the type of marginals can be taken into account.

Suggestion

We suggest to use the dependence function $\Lambda(F(x), G(y))$ introduced by Sibuya (1960), such that

$$H(x, y) = \Lambda(F(x), G(y))F(x)G(y) \tag{1}$$

for all $(x, y) \in \Re^2$.

We will refer to $\Lambda(F(x), G(y))$ as *Sibuya's dependence function, (SDF)*.

Remark. One can see that the marginal behavior is taken into account in (1).

Example 2

We will now show $\Lambda_1(x,y)$ and $\Lambda_2(x,y)$, the SDF of $H_1(x,y)$ and $H_2(x,y)$ given in Example 1. We obtain

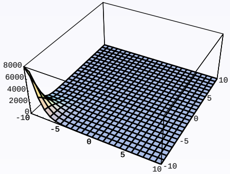
$$\Lambda_1(x,y) = \frac{[1+\exp(-x)][1+\exp(-y)]}{1+\exp(-x)+\exp(-y)} = 1 + \frac{e^{-(x+y)}}{1+e^{-x}+e^{-y}} > 1,$$

$$\text{for } x,y \in (-\infty,\infty).$$

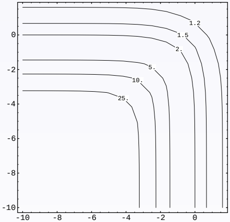
which is different than

$$\Lambda_2(x,y) = \begin{cases} \dfrac{1}{1-\frac{1-x}{2}e^{-y}}, & \text{if } (x,y) \in [-1,1]\times[0,\infty); \\ 1, & \text{if } (x,y) \in (1,\infty)\times[0,\infty); \\ 0, & \text{elsewhere.} \end{cases}$$

Graphs - Example 2

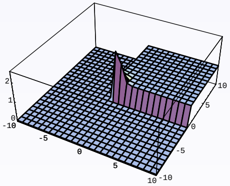


(a) $\Lambda_1(x,y) = \beta$ where $\beta = 1.2, 1.5, 2, 5, 10, 25.$

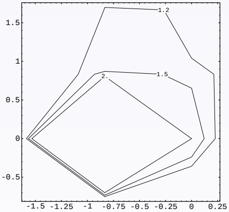


(b) 3-d plot.

Graphs - Example 2



(c) $\Lambda_2(x, y) = \beta$ where $\beta = 1.2$, $1.5, 1.9, 2.5$.



(d) 3-d plot.

Remark 2

As a conclusion, the SDF's of two distributions with the same copula function (as in Example 1) and different marginal distributions will differ. It is due to the fact that SDF is determined by any of the following equivalent ratios

$$\Lambda(x,y) = \frac{C(F(x),G(y))}{F(x)G(y)} = \frac{H(x,y)}{F(x)G(y)} = \frac{P(X \le x, Y \le y)}{P(X \le x)P(Y \le y)}.$$

Therefore, two distributions $H_i(x,y)$ with marginals $F_i(x)$, and $G_i(y)$, $(i = 1,2)$, and with coinciding copulas will have the same SDF if and only if $F_1(x)G_1(y) = F_2(x)G_2(y)$.

Interpretation of the SDF - 1

It is possible to see from

$$H(x,y) = \Lambda(F(x),G(y))F(x)G(y)$$

that $\Lambda(F(x),G(y))$ is a scale distance between independence and the genuine dependence structure, represented by $H(x,y)$. For each fixed point $(x_0,y_0) \in \Re^2$ the quantity $\Lambda(F(x_0),G(y_0))$ is the homothety which transfers the virtual independence property (given by $F(x_0)G(y_0)$) into the true dependence acting at the point (x_0,y_0).

Interpretation of the SDF - 2

Another look at the

$$H(x, y) = \Lambda(F(x), G(y))F(x)G(y)$$

allows us to say that the joint distribution $H(x, y)$ is decomposed in 3 multiplicative factors: the influence of X itself (given by $F(x)$), the influence of Y itself (given by $G(y)$) and the influence of the interaction effect between the random variables (given by the SDF Λ). **This fact differs the SDF function from copulas**.

Properties

Here we present some properties of SDF according to the original paper of Sibuya (1960)

- **(P1)** $\Lambda(F(x), G(y)) = 1$, if and only if X and Y are independent random variables;
- **(P2)** The *SDF* is *ordinally invariant*. That is, if $\phi(x)$ and $\psi(y)$ are monotone non-decreasing functions, the *SDF* of (X, Y) and $(\phi(X), \psi(Y))$ are the same;
- **(P3)** $max\left\{0, \frac{G(y)+F(x)-1}{F(x)G(y)}\right\} \leq \Lambda(F(x), G(y)) \leq min\left\{\frac{1}{F(x)}, \frac{1}{G(y)}\right\}.$

Properties - Characterization

Characterization Lemma. The *SDF* $\Lambda_H(x,y)$ of any continuous joint distribution $H(x,y)$ with continuous marginals $F(x)$ and $G(y)$ is related to the *SDF* $\Lambda_C(u,v)$ of the associated copula $C(u,v)$ by the relations

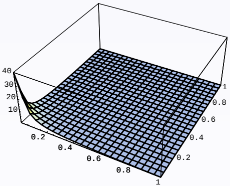
$$\Lambda_H(F(x),G(y)) = \Lambda_C(F(x),G(y))$$

and

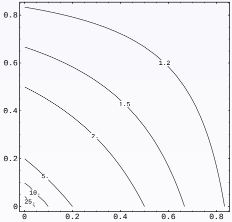
$$\Lambda_C(u,v) = \Lambda_H(F(F^{-1}(u)),G(G^{-1}(v))).$$

Example - Graph

Here we present the graph of the *SDF* related to the copula $C(u,v) = \frac{uv}{u+v-uv}$ (Example 1)



(e) $\Lambda_3(u,v) = \beta$ where $\beta = 1.2, 1.5, 2, 5, 10, 25.$



(f) 3-d plot.

Properties - Alternative Representation

Theorem. The following alternative representation of the *SDF* is based on *conditional probabilities*, i.e.

$$\Lambda(F(x), G(y)) =$$

$$= 1 + \frac{[P(Y \leq y|X \leq x) - P(Y \leq y|X > x)][1 - F(x)]}{G(y)} \qquad (2)$$

$$= 1 + \frac{[P(X \leq x|Y \leq y) - P(X \leq x|Y > y)][1 - G(y)]}{F(x)}.$$

Example 3

Consider the FGM family of distributions given by

$$H(x,y) = F(x)G(y)[1 + \theta(1 - F(x))(1 - G(y))],$$

where $-1 \le \theta \le 1$. Using the definition of the SDF and the above theorem, one gets

$$\theta = \frac{P(Y \le y|X \le x) - P(Y \le y|X > x)}{G(y)(1 - G(y))}.$$

Note that since the denominator is independent of x, the numerator must also be independent since θ is constant. From the above relation follows

$$-\frac{1}{4} \le P(Y \le y|X \le x) - P(Y \le y|X > x) \le \frac{1}{4},$$

i.e. FGM family serves to describes a weak dependence only.

Properties - Distribution

For a joint distribution $H(x, y)$ with marginals $F(x)$ and $G(y)$, we define the distribution $L_H(X, Y)(w)$ as follows

$$L_H(X, Y)(w) = P(\Lambda(F(X), G(Y)) \leq w) = P\left\{\frac{H(X, Y)}{F(X)G(Y)} \leq w\right\},$$

for $w > 0$.

Proposition: Let H_L, H_Π and H_U be the joint distribution functions when X and Y are countermonotonic, independent and comonotonic random variables, respectively. Related probabilistic properties of the SDF are given bellow.

$$(i) \quad L_{H_L}(X,Y)(w) = I_{[w\geq 0]}; \tag{3}$$

$$(ii) \quad L_{H_\Pi}(X,Y)(w) = I_{[w\geq 1]}; \tag{4}$$

$$(iii) \quad L_{H_U}(X,Y)(w) = \begin{cases} 0, & \text{if } 0 < w < 1; \\ 1 - \frac{1}{w}, & \text{if } w \geq 1; \end{cases} \tag{5}$$

$$(iv) \quad L_{H_U}(X,Y)(w) \leq L_{H_\Pi}(X,Y)(w) \leq L_{H_L}(X,Y)(w) \quad \text{for} \quad w > 0. \tag{6}$$

Properties - Monotone Transformations

Given arbitrary functions $\alpha(x)$ and $\beta(y)$ on the support of the continuous random variables X and Y, respectively, such that the inverses $\alpha^{-1}(.)$ and $\beta^{-1}(.)$ do exist, we have:
(i) If both $\alpha(x)$ and $\beta(y)$ are **increasing functions then**

$$\Lambda_{\alpha(X)\beta(Y)}(x,y) = \Lambda_{XY}(\alpha^{-1}(x), \beta^{-1}(y));$$

ii) *If* $\alpha(x)$ *is a* **decreasing function,** $\beta(y)$ *is an* **increasing function** and $\Theta_F(x) = \frac{F(x)}{\overline{F}(x)}$, *then*

$$\Lambda_{\alpha(X)\beta(Y)}(x,y) = 1 + \Theta_F(\alpha^{-1}(x))[1 - \Lambda_{XY}(\alpha^{-1}(x), \beta^{-1}(y))]$$

Similar results can be obtained in the other possible cases.

Another Characterization for the SDF

Characterization Theorem. A function $\Lambda(.,.)$ is a valid *SDF* if and only if

$$\ln \Lambda(F(X), G(Y)) = S + W + T,$$

where S and W are unit exponential random variables and T is a random variable such that e^T is Kendall distributed.

Remark 3

Let $H(x,y)$ be positive quadrant dependent, i.e. $(H(x,y) \geq F(x)G(y))$ for all $(x,y) \in \Re^2)$.

The Decomposition Theorem shows that the sum of positively correlated unit exponential variates S and W "compensates" the Kendall distributed random variable to Sibuya distributed one on the logarithmic scale.

Properties - Lower Bound for the Expectation

Since $\Lambda(F(X), G(Y))$ is always positive, an **lower bound** for $E(\Lambda(F(X), G(Y)))$ is 0. Using the Decomposition Theorem we can get **another lower bound for the expectation of** $\Lambda(F(X), G(Y))$, i.e.

$$E(\Lambda_{XY}(X, Y)) \geq e^{\kappa+2},$$

where

$$\kappa =: \int_0^1 \ln t \; dK(t).$$

Properties - Other Properties

- **Sample Properties**: There are several properties of the empirical Sibuya's function, such as almost surely convergence under the independence hypothesis.
- **Informative Lower Bound for the Expectation:** Conditions to obtain a nonzero lower bound can be derived.

Advantages and disadvantages of using the SDF

Advantages.

- There is a simple procedure to estimate the *SDF*, based on empirical step functions. Most of copulas estimation are *heuristic or Ad-Hoc* procedures.
- We can draw dependence lines $A_\beta(x,y) = \{(x,y) : \Lambda(F(x), G(y)) = \beta, \beta \in \Re^+\}$ which display local dependence structure on $\Re^2$. In each line we have the set of points (x,y) so that its scale distance towards independence is a fixed number β.
- The *SDF* gives us an idea about association effect between variables X and Y and its contribution to the dependence structure.

As we can see in the figures copulas does not show these patterns explicitly. **Therefore copulas oversimplify the analysis of the dependence structure.**

Advantages and disadvantages of using the SDF

Disadvantages

- Up to our knowledge, the *SDF* is is still restricted to bivariate scenarios.
- The copula function is well known. Its meaning and how to use it in data analysis is already a well developed field in statistics.
- Copulas are simpler to deal with.

References

- Fernandez, M. and Kolev, N. (2007). Bivariate density characterization by geometry of marginals. Forthcoming in *Economic Quality Control* **22**.
- Nelsen, R. (2006). *An Introduction to Copulas*. Springer, New York.
- Sibuya, M. (1960). Bivariate extreme statistics. *Annals of the Institute of Statistical Mathematics* **11**, 195-210.