Shifted Multivariate Asymmetric Laplace Distribution

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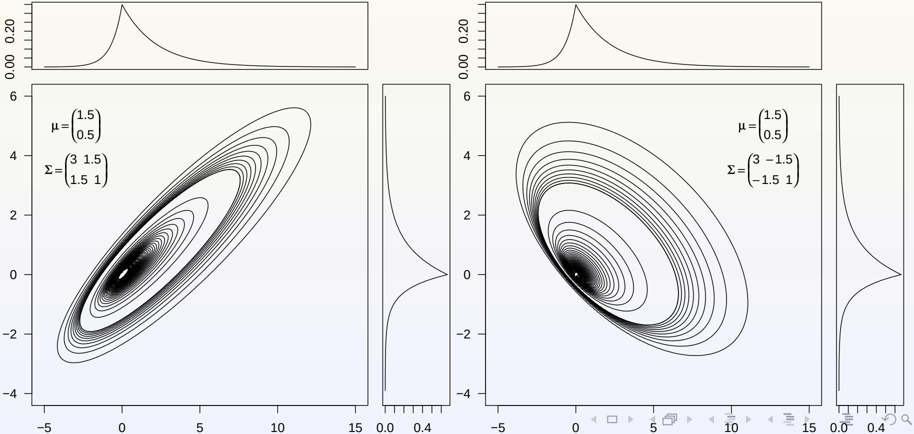
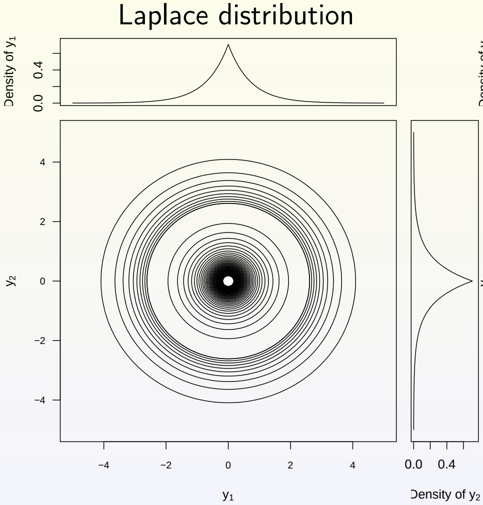
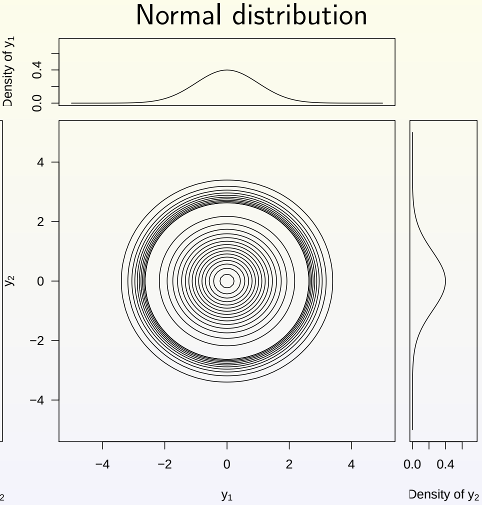
Laplace distribution

Let $\boldsymbol{\mu}$ be a p-vector and $\boldsymbol{\Sigma}$ a positive definite $p \times p$-matrix. The characteristic function of p-variate Laplace distribution is

$$\varphi_{\mathbf{x}}(\mathbf{t}) = \frac{1}{1 - i\,\mathbf{t}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}}, \text{ shortly } \mathbf{x} \sim L_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}).$$




Laplace distribution
Density of y_1
0.0
0.4
y_2
4
2
0
−2
−4
y_1
Density of y_2


Normal distribution
Density of y_1
0.0
0.4
y_2
4
2
0
−2
−4
y_1
Density of y_2

- Linear transformations in the same family:
  $\mathbf{x} \sim L_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$,
  $\mathbf{A}$: $q \times p$-matrix ($q < p$, full rank)
  $\Rightarrow \mathbf{A}\mathbf{x} \sim L_q(\mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T)$.
- Easy to generate:

$$\begin{aligned} W &\sim Exp(1), \\ \mathbf{z} &\sim N_p(\mathbf{0}, \boldsymbol{\Sigma}), \\ \mathbf{x} &= \boldsymbol{\mu} \cdot W + \sqrt{W} \cdot \mathbf{z} \text{ has Laplace distribution } L_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \end{aligned}$$

- Moments:

$$\begin{aligned} E\mathbf{x} &= \boldsymbol{\mu}, \\ D\mathbf{x} &= \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^T, \\ \overline{m}_3(\mathbf{x}) &= 2\boldsymbol{\mu} \otimes \boldsymbol{\mu}\boldsymbol{\mu}^T + \mathrm{vec}\,\boldsymbol{\Sigma} \cdot \boldsymbol{\mu}^T + \boldsymbol{\mu} \otimes \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \otimes \boldsymbol{\mu}. \end{aligned}$$

Generalizing Laplace distribution

Parameter $\boldsymbol{\mu}$ controls both location and skewness. Shifting by constant $\mathbf{a}$ leads to chf

$$\varphi_{\mathbf{x}+\mathbf{a}}(\mathbf{t}) = \frac{e^{i\mathbf{t}^T\mathbf{a}}}{1 - i\,\mathbf{t}^T\boldsymbol{\mu} + \frac{1}{2}\mathbf{t}^T\boldsymbol{\Sigma}\mathbf{t}}.$$

Definition

Let $\mathbf{x} \sim L_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{a}$ be a p-vector. Then $\mathbf{y} = \mathbf{x} + \mathbf{a}$ has shifted multivariate asymmetric Laplace distribution,

$$\mathbf{y} \sim L_p(\mathbf{a}, \boldsymbol{\mu}, \boldsymbol{\Sigma}).$$

- $\mathbf{a} = \mathbf{0}$ – (ordinary) Laplace distribution
- $\mathbf{a} = -\boldsymbol{\mu}$ – distribution of centered vector

Properties

Some properties can be easily taken over from Laplace distribution:

- characteristic function,
- distribution function & density function
- simulation algorithm,
- central moments remain the same,
- linear transformations are in the same family

$$\mathbf{y} \sim L_p(\mathbf{a}, \boldsymbol{\mu}, \boldsymbol{\Sigma}),\ \mathbf{A} : q \times p \Rightarrow \mathbf{Ay} \sim L_q(\mathbf{Aa}, \mathbf{A}\boldsymbol{\mu}, \mathbf{A}\boldsymbol{\Sigma}\mathbf{A}^T).$$

Parameter estimation

Method of moments: express parameters through moments.
Moments are

$$\begin{aligned} E\mathbf{y} &= \boldsymbol{\mu} + \mathbf{a}, \\ D\mathbf{y} &= \boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}^T, \\ \overline{m}_3(\mathbf{y}) &= 2\boldsymbol{\mu} \otimes \boldsymbol{\mu}\boldsymbol{\mu}^T + \operatorname{vec} \boldsymbol{\Sigma} \cdot \boldsymbol{\mu}^T + \boldsymbol{\mu} \otimes \boldsymbol{\Sigma} + \boldsymbol{\Sigma} \otimes \boldsymbol{\mu}. \end{aligned}$$

Three equations, three unknowns $(\mathbf{a}, \boldsymbol{\mu}, \boldsymbol{\Sigma})$.

Method of moments

$$\mathbf{1}_{p\times p} \star \overline{m}_3(\mathbf{y}) = \left(S - M^2\right)\boldsymbol{\mu} + 2M \cdot D\mathbf{y}\mathbf{1}_p.$$

M – sum of parameter $\boldsymbol{\mu}$,
S – sum of dispersion matrix $D\mathbf{y}$,
H – sum of third central moment $\overline{m}_3(\mathbf{y})$.
Summing the elements on both sides leads to cubic equation

$$M^3 - 3MS + H = 0,$$

solution is

$$M = -\frac{z}{4} - \frac{S}{z} - \frac{i\sqrt{3}}{2}\left(\frac{z}{2} - \frac{2S}{z}\right), \qquad z = \sqrt[3]{-4H + 4\sqrt{H^2 - 4S^3}}.$$

$$\begin{aligned}
\boldsymbol{\mu} &= \frac{1}{S - M^2}\left(\mathbf{1}_{p\times p} \star \overline{m}_3(\mathbf{y}) - 2M \cdot D\mathbf{y}\mathbf{1}_p\right), \\
\boldsymbol{\Sigma} &= D\mathbf{y} - \boldsymbol{\mu}\boldsymbol{\mu}^T, \\
\mathbf{a} &= E\mathbf{y} - \boldsymbol{\mu}.
\end{aligned}$$

Works well if the distribution is symmetric ($\boldsymbol{\mu} = \mathbf{0}$) .
In the presence of skewness it is possible to get complex estimate for M or estimate for $\boldsymbol{\Sigma}$ that is not positive definite from some of the samples.
Solutions?

- use only real part of M (unstable);
- 'correct' the estimates so that M is real and $\sigma_{ii} > 0$ (multimodal distributions).

Method of modes

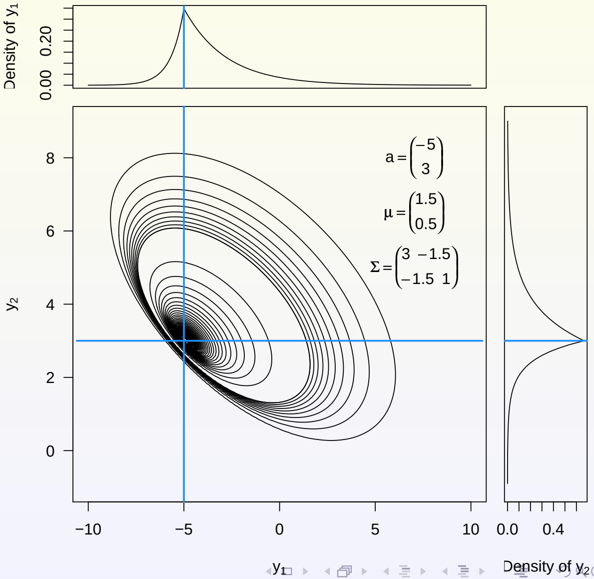
Modal value in $\mathbf{a}$.

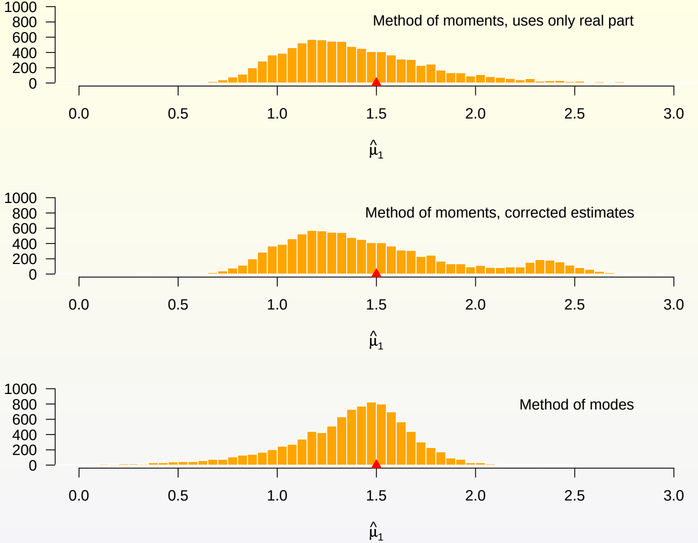
Estimates:

$\mathbf{a}$ – modal value,

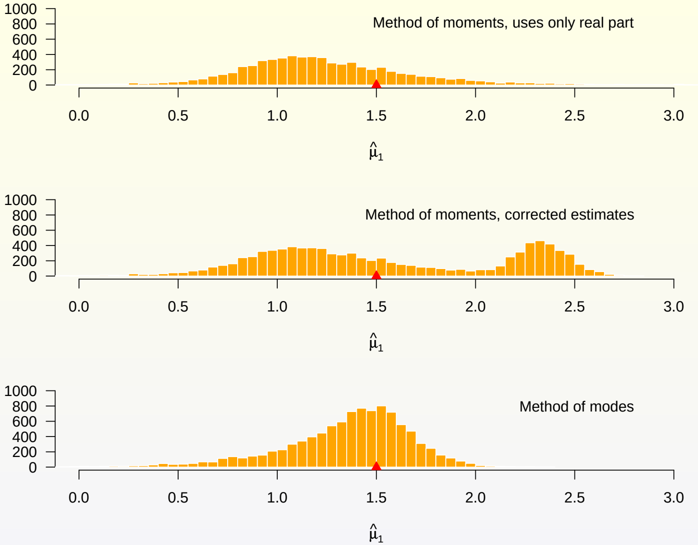
$\boldsymbol{\mu} = E\mathbf{y} - \mathbf{a}$,

$\boldsymbol{\Sigma} = D\mathbf{y} - \boldsymbol{\mu}\boldsymbol{\mu}^T$.



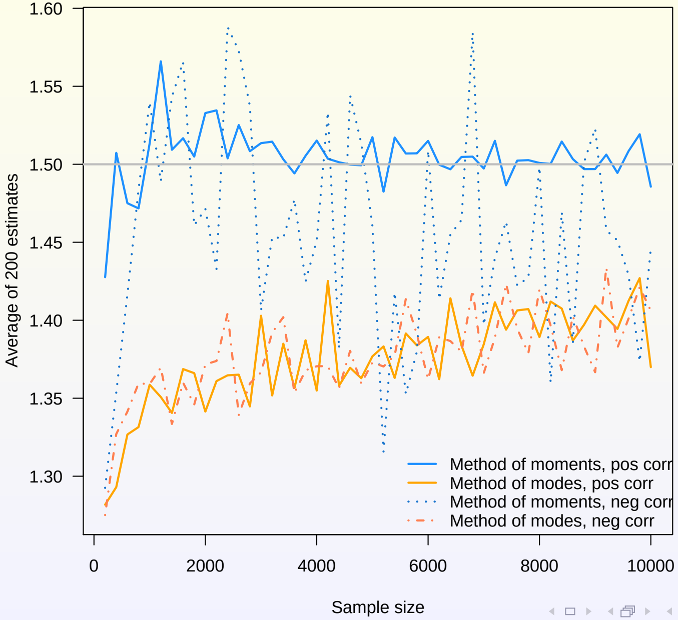


$$\mathbf{a} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}, \ \boldsymbol{\mu} = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} 3 & 1.5 \\ 1.5 & 1 \end{pmatrix}$$



$$\mathbf{a} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}, \ \boldsymbol{\mu} = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} 3 & -1.5 \\ -1.5 & 1 \end{pmatrix}$$

Convergence of estimates of μ_1. 2 parameter sets, σ_{12} differs: $\sigma_{12} = 1.5$ (blue) vs $\sigma_{12} = -1.5$ (orange).



Thank you for your attention!