“Weak” Stochastic Orderings and Dependence Measures

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Positive Quadrant Dependence

$$\boxed{\mathrm{PQD}(X,Y)}$$

(X,Y) has *positive quadrant dependence* if

$$C(u,v) \geq uv$$

for all (u,v) in the unit square.

Tail Monotonicity

$$\boxed{\mathrm{RTI}(Y|X)}$$

Y is *right tail increasing* in X if

$$\mathrm{P}[Y > y | X > x]$$

is a nondecreasing function of x for all y.

Right Tail Decreasing, *Left Tail Increasing*, and *Left Tail Decreasing* are defined similarly.

Stochastic Monotonicity

$$\boxed{\mathrm{SI}(Y|X)}$$

Y is *stochastically increasing* in X if

$$\mathsf{P}[Y > y|X = x]$$

is a nondecreasing function of x for all y.

This property is also called *Positive Regression Dependence*.

Stochastically Decreasing and *Negative Regression Dependence* are defined similarly.

Corner Set Monotonicity

$$\boxed{\mathrm{RCSI}(X, Y)}$$

(X, Y) is *right corner set increasing* if

$$\mathsf{P}[X > x, Y > y | X > x', Y > y']$$

is a nondecreasing function of x' and y'.

Right Corner Set Decreasing, *Left Corner Set Increasing*, and *Left Corner Set Decreasing* are defined similarly.

Total Positivity

$$\boxed{\mathsf{TP}_2(X,Y)}$$

(X,Y) has *total positive dependence of order 2* if

$$\begin{vmatrix} h(x,y) & h(x,y') \\ h(x',y) & h(x',y') \end{vmatrix} \geq 0$$

whenever $x \leq x'$ and $y \leq y'$.

This property is also called *Positive Likelihood Ratio Dependence*.

When the inequality is reversed, (X,Y) is *Reverse Regular of Order 2*.

Some Relationships

$$
\begin{tikzcd}
 & \mathrm{TP}_2(X,Y) \arrow[dl] \arrow[dr] & \\
\mathrm{SI}(Y|X) \arrow[dr] & & \mathrm{RCSI}(X,Y) \arrow[dl] \\
 & \mathrm{RTI}(Y|X) \arrow[d] & \\
 & \mathrm{PQD}(X,Y) &
\end{tikzcd}
$$

Bivariate Probability Integral Transform

Let H_1 and H_2 be bivariate distribution functions with common continuous marginal distribution functions F and G.

Let X and Y be random variables whose joint distribution function is H_2.

Let $\langle H_1|H_2\rangle(X,Y)$ denote the random variable $H_1(X,Y)$.

The H_2 *distribution function of* H_1 is

$$\begin{aligned}(H_1|H_2)(t) &= \Pr[\langle H_1|H_2\rangle(X,Y) \le t] \\ &= \mu_{H_2}(\{(x,y) \in \Re^2 | H_1(x,y) \le t\}), t \ge 0.\end{aligned}$$

Kendall Distribution Function

The *Kendall random variable* associated with a random vector $\mathbf{X}$ is $T = H(\mathbf{X})$, where $H(\mathbf{x})$ is the joint c.d.f. of $\mathbf{X}$.

The *Kendall distribution function* of the random vector $\mathbf{X}$ is the cumulative distribution function of the Kendall random variable T.

In the bivariate case, $K(t) = (H|H)(t) = (C|C)(t)$.

Kendall Ordering

(Capéraà, *et al.*, 1997) The *Kendall stochastic ordering* of continuous random vectors (X_1, Y_1) and (X_2, Y_2), with distribution functions H_1 and H_2, respectively, is defined as:

$$(X_1, Y_1) \prec_K (X_2, Y_2) \text{ if and only if } H_1(X_1, Y_1) \prec_{st} H_2(X_2, Y_2),$$

where $\prec_{st}$ denotes stochastic ordering. That is,

$$(X_1, Y_1) \prec_K (X_2, Y_2) \iff K_1(t) \geq K_2(t) \quad \forall t \in \Re.$$

Since the Kendall distribution functions depend only on the copulas C_1 and C_2, the notation $(X_1, Y_1) \prec_K (X_2, Y_2)$ is sometimes abbreviated as $C_1 \prec_K C_2$.

Positive Kendall Dependence

$$\boxed{\mathsf{PKD}(X,Y)}$$

(X,Y) has *positive Kendall dependence* if

$$C \succ_K \Pi,$$

that is, if

$$K_C(t) \le K_{\Pi}(t) = t - t\ln t.$$

PQD vs. PKD

PQD means

$$C(u,v) \geq uv \quad \forall (u,v) \in I^2.$$

This is equivalent to

$$C(u,v) \leq t \Rightarrow uv \leq t \quad \forall t \in I.$$

This implies that, $\forall t \in I$, both

$$\mathsf{P}_C[C(U,V) \leq t] \leq \mathsf{P}_C[UV \leq t]$$

and

$$\mathsf{P}_\Pi[C(U,V) \leq t] \leq \mathsf{P}_\Pi[UV \leq t].$$

PQD vs. PKD, cont.

That is, PQD implies that, $\forall t \in I$, both

$$(C|C)(t) \leq (\Pi|C)(t)$$

and

$$(C|\Pi)(t) \leq (\Pi|\Pi)(t).$$

On the other hand, PKD means

$$(C|C)(t) \leq (\Pi|\Pi)(t) \quad \forall t \in I.$$

PQD vs. PKD, cont.

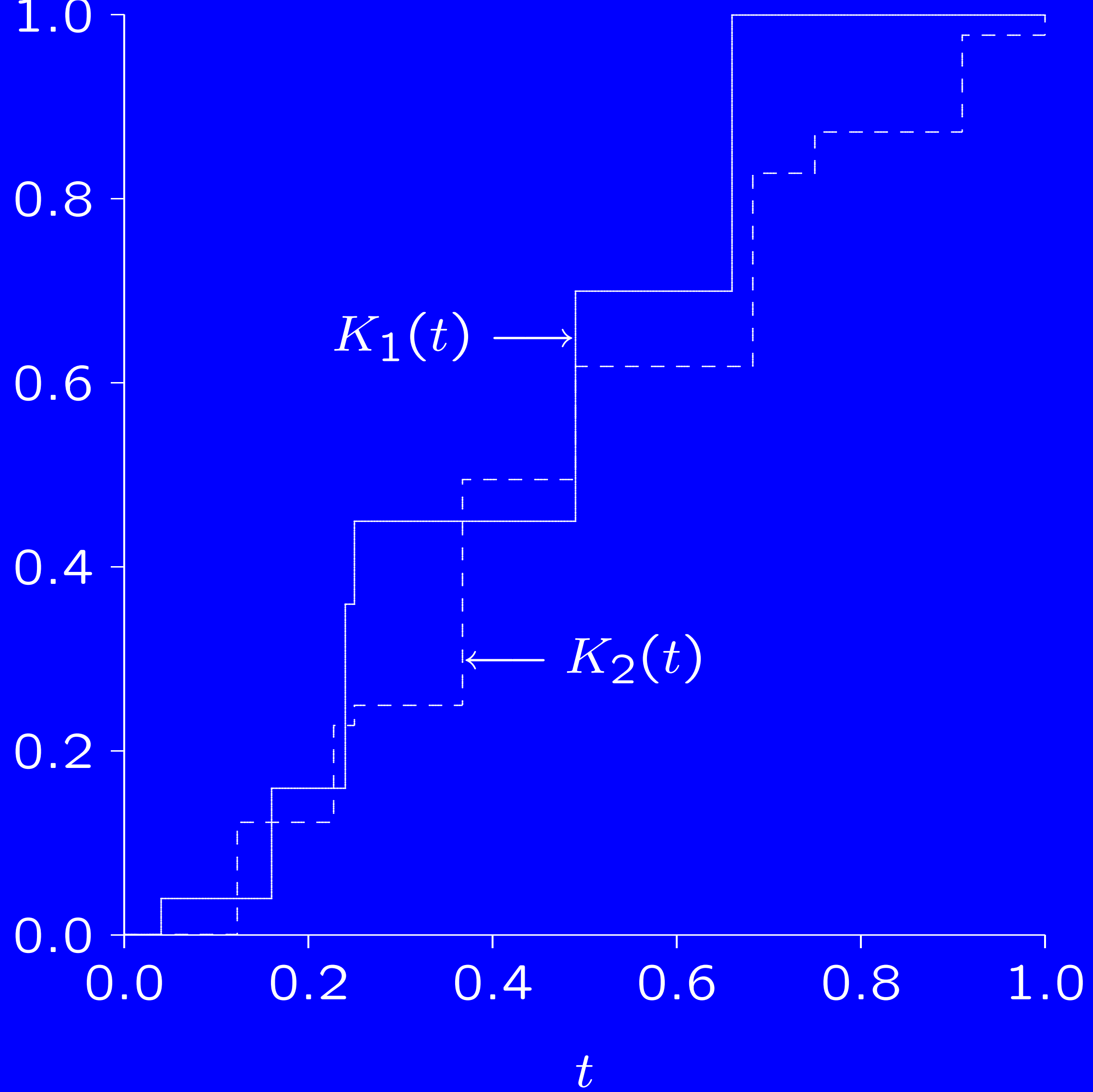
So, a sufficient condition for PQD to imply PKD is

$$(\Pi|C)(t) \le (C|\Pi)(t) \quad \forall t \in I,$$

or equivalently,

$$\mathsf{P}_C[UV \le t] \le \mathsf{P}_\Pi[C(U,V) \le t] \quad \forall t \in I.$$

Kendall Incomparability



Quadrant Ordering

Let (X_1, Y_1) and (X_2, Y_2) be random vectors with copulas C_1 and C_2. Let μ be a probability measure on the unit square. Define

$$(X_1, Y_1) \prec_\mu (X_2, Y_2) \quad \text{and} \quad C_1 \succ_\mu C_2 \quad \text{if}$$

$$\mathsf{p}_\mu(C_1, C_2) = \mathsf{P}_\mu[C_1(U,V) > C_2(U,V)] > 0.5.$$

If there is a positive μ-probability that $C_1(U,V) = C_2(U,V)$, then $\mathsf{p}_\mu(C_1, C_2)$ may be replaced by either

$$\mathsf{p}^*_\mu(C_1, C_2) = \mathsf{P}_\mu[C_1(U,V) > C_2(U,V)] + \frac{1}{2}\mathsf{P}_\mu[C_1(U,V) = C_2(U,V)]$$

or

$$\mathsf{p}^{**}_\mu(C_1, C_2) = \frac{\mathsf{P}_\mu[C_1(U,V) > C_2(U,V)]}{1 - \mathsf{P}_\mu[C_1(U,V) = C_2(U,V)]}.$$

μ-Quadrant Dependence

Let (X, Y) be a random vector with copula C. Let μ be a probability measure on the unit square. Define

$$\mathsf{QD}_\mu(X, Y) = 2\mathsf{p}_\mu(C, \Pi) - 1.$$

(X, Y) has Positive μ-Quadrant Dependence if

$$\mathsf{QD}_\mu(X, Y) > 0.$$

Discrete Example

		Y_1			
	pdf	1	2	3	
	1	0.20	0.00	0.10	0.30
X_1	2	0.05	0.20	0.05	0.30
	3	0.20	0.00	0.20	0.40
		0.45	0.20	0.35	1.00

		Y_1			
	cdf	1	2	3	
	1	0.20	0.20	0.30	
X_1	2	0.25	0.45	0.60	
	3	0.45	0.65	1.00	

Pearson's $\rho = 0.149$.

Kendall's $\tau = 0.139$.

Now suppose that X_2 and Y_2 have the same marginals as before, but that they are independent.

	pdf	Y_2 = 1	2	3	
X_2	1	0.135	0.060	0.105	0.300
	2	0.135	0.060	0.105	0.300
	3	0.180	0.080	0.140	0.400
		0.450	0.200	0.350	1.000

	cdf	Y_2 = 1	2	3
X_2	1	0.135	0.195	0.300
	2	0.270	0.390	0.600
	3	0.450	0.650	1.000

We now compare the two cumulative distributions,

Y_1

X_1 cdf	1	2	3
1	0.20	0.20	0.30
2	0.25	0.45	0.60
3	0.45	0.65	1.00

Y_2

X_2 cdf	1	2	3
1	0.135	0.195	0.300
2	0.270	0.390	0.600
3	0.450	0.650	1.000

and note the following pattern:

Y

X cdf	1	2	3
1	1	1	0.5
2	0	1	0.5
3	0.5	0.5	0.5

Compare this pattern to the original joint probability distribution:

		Y		
	cdf	1	2	3
	1	1	1	0.5
X	2	0	1	0.5
	3	0.5	0.5	0.5

		Y_1			
	pdf	1	2	3	
	1	0.20	0.00	0.10	0.30
X_1	2	0.05	0.20	0.05	0.30
	3	0.20	0.00	0.20	0.40
		0.45	0.20	0.35	1.00

and find:

$\mathsf{p}^*_C(C,\Pi)$	QD^*_C	$\mathsf{p}^{**}_C(C,\Pi)$	QD^{**}_C
0.675	0.350	0.889	0.778

Compare this pattern to the independent joint probability distribution:

	cdf	Y = 1	2	3	
	1	1	1	0.5	
X	2	0	1	0.5	
	3	0.5	0.5	0.5	

	pdf	Y_2 = 1	2	3	
	1	0.135	0.060	0.105	0.300
X_2	2	0.135	0.060	0.105	0.300
	3	0.180	0.080	0.140	0.400
		0.450	0.200	0.350	1.000

and find:

$\mathsf{p}^*_\sqcap(C,\sqcap)$	$\mathsf{QD}^*_\sqcap$	$\mathsf{p}^{**}_\sqcap(C,\sqcap)$	$\mathsf{QD}^{**}_\sqcap$
0.560	0.120	0.654	0.308

Compare this pattern to the uniform joint probability distribution:

	cdf	Y 1	2	3	
	1	1	1	0.5	
X	2	0	1	0.5	
	3	0.5	0.5	0.5	

	pdf	Y_2 1	2	3	
	1	0.111	0.111	0.111	0.333
X_2	2	0.111	0.111	0.111	0.333
	3	0.111	0.111	0.111	0.333
		0.333	0.333	0.333	1.000

and find:

$\mathrm{p}^*_\mu(C,\Pi)$	QD^*_μ	$\mathrm{p}^{**}_\mu(C,\Pi)$	QD^{**}_μ
0.611	0.222	0.750	0.500

Compare this pattern to the probability distribution for the copula M:

		Y		
	cdf	1	2	3
	1	1	1	0.5
X	2	0	1	0.5
	3	0.5	0.5	0.5

		Y_2			
	pdf	1	2	3	
	1	0.30	0.00	0.00	0.30
X_2	2	0.15	0.15	0.00	0.30
	3	0.00	0.05	0.35	0.40
		0.45	0.20	0.35	1.00

and find:

$\mathsf{p}^*_M(C,\Pi)$	QD^*_M	$\mathsf{p}^{**}_M(C,\Pi)$	QD^{**}_M
0.650	0.300	0.750	0.500

Compare this pattern to the probability distribution for the copula W:

	Y			
	cdf	1	2	3
	1	1	1	0.5
X	2	0	1	0.5
	3	0.5	0.5	0.5

	Y_2				
	pdf	1	2	3	
	1	0.00	0.00	0.30	0.30
X_2	2	0.05	0.20	0.05	0.30
	3	0.40	0.00	0.00	0.40
		0.45	0.20	0.35	1.00

and find:

$\mathsf{p}^*_W(C,\Pi)$	QD^*_W	$\mathsf{p}^{**}_W(C,\Pi)$	QD^{**}_W
0.575	0.150	0.800	0.600

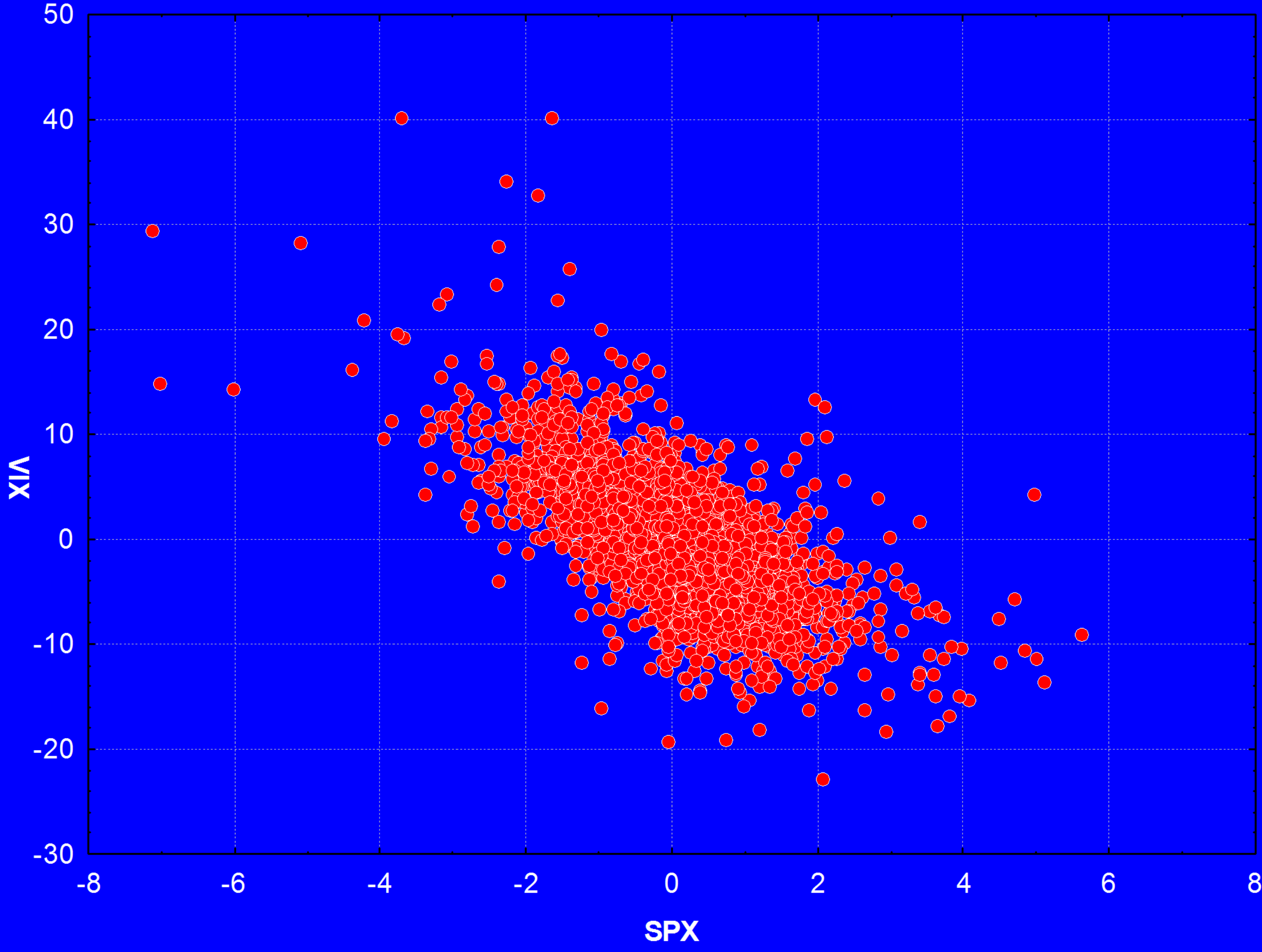
VIX/SPX Data Set

SPX:	the Standard & Poors 500 index in US$
VIX:	the Chicago Board of Trades index for implied volatility
Period:	from 2 January 1990 through 31 December 2004
Content:	3784 daily observations

Pre-processing

ARMA:	autoregressive and moving average components
GARCH:	heteroscadascity or serial correlation in the variances
GJR-GARCH: TGARCH:	differences between negative and positive shocks
Structural Change:	varying volatility from one time period to another
Trend Shifting:	shift in the trend of conditional volatility

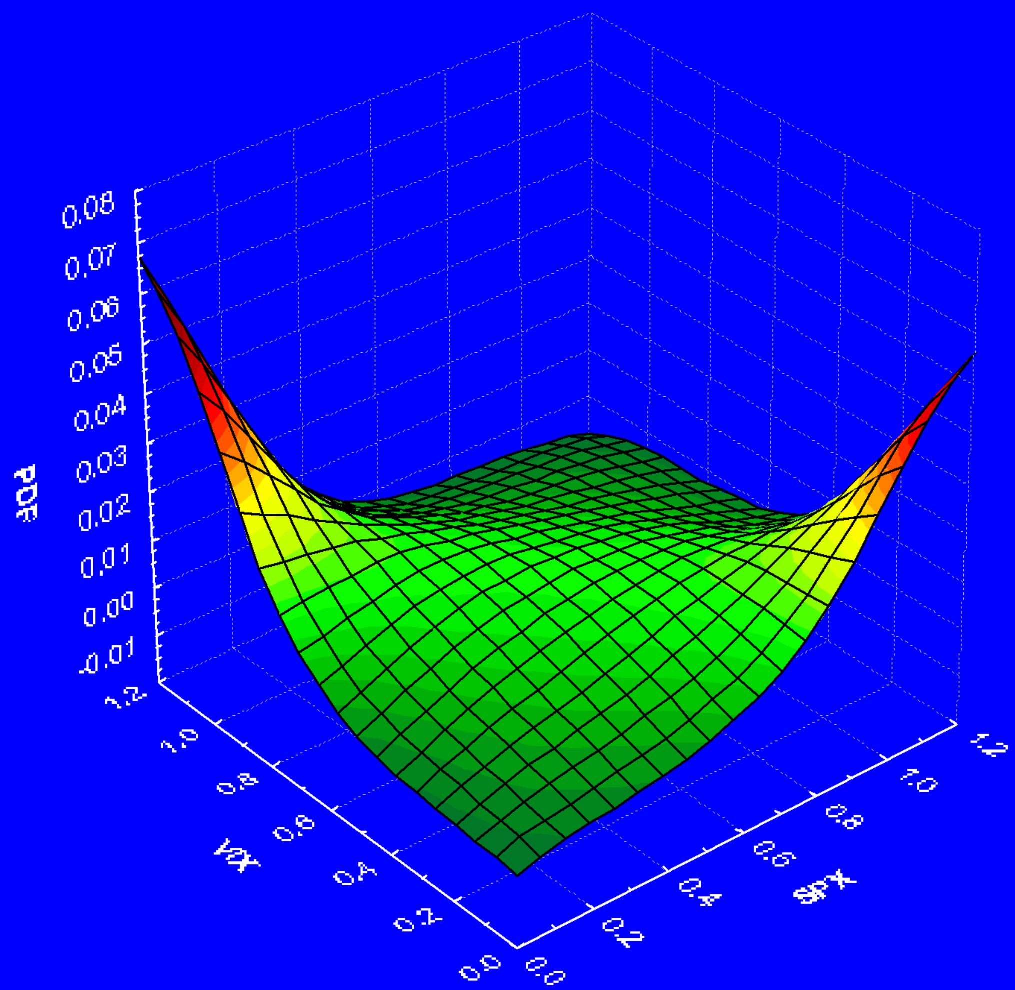
VIX/SPX Residuals



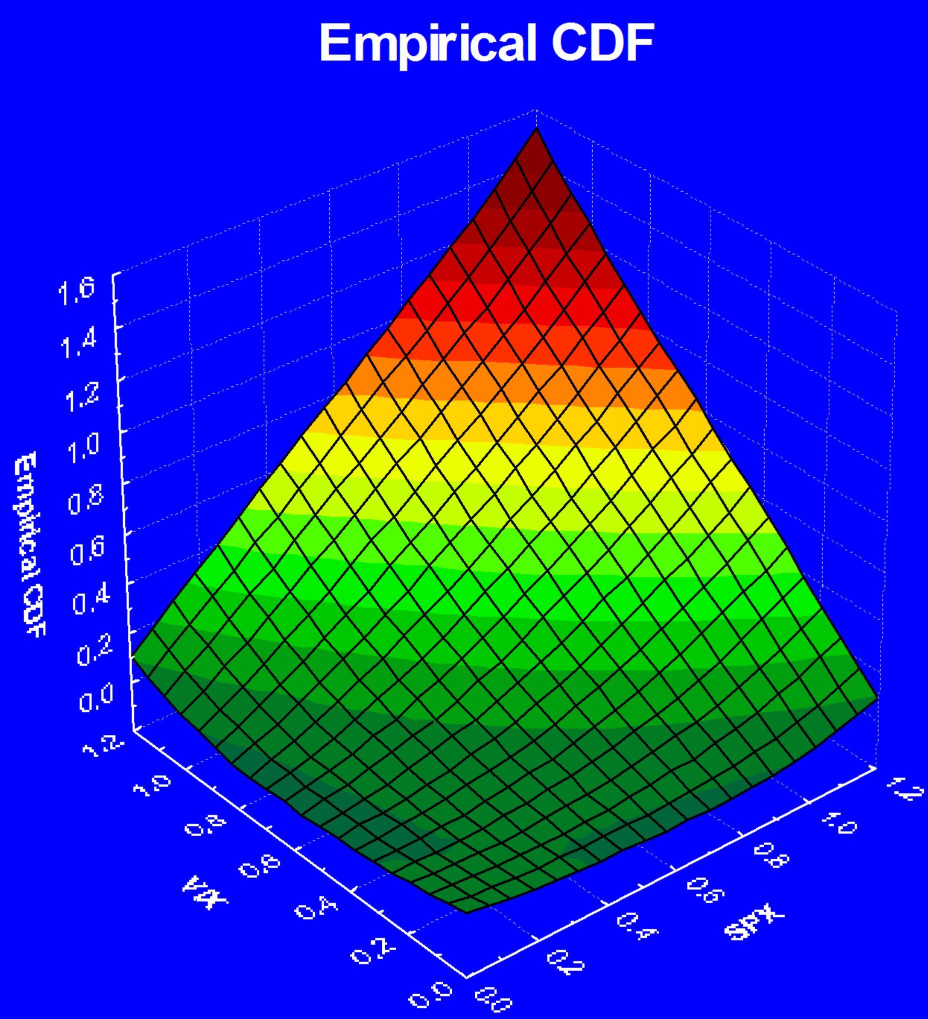
Correlations

Pearson:	-0.661
Kendall:	-0.476
Spearman:	-0.653

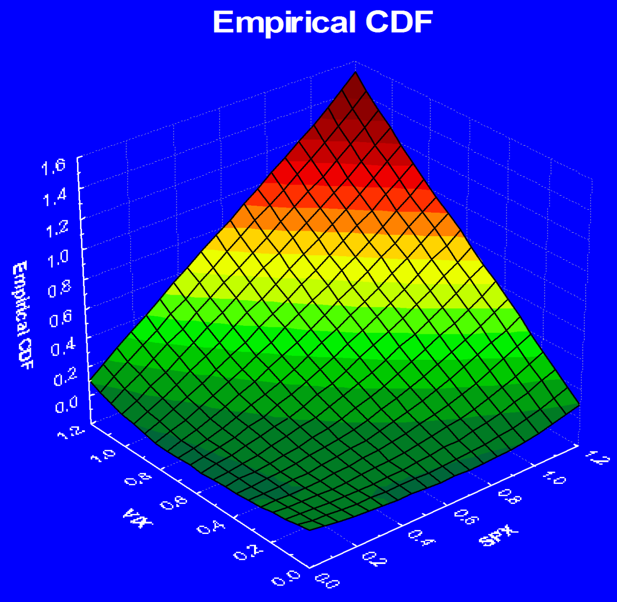
VIX/SPX Empirical PDF

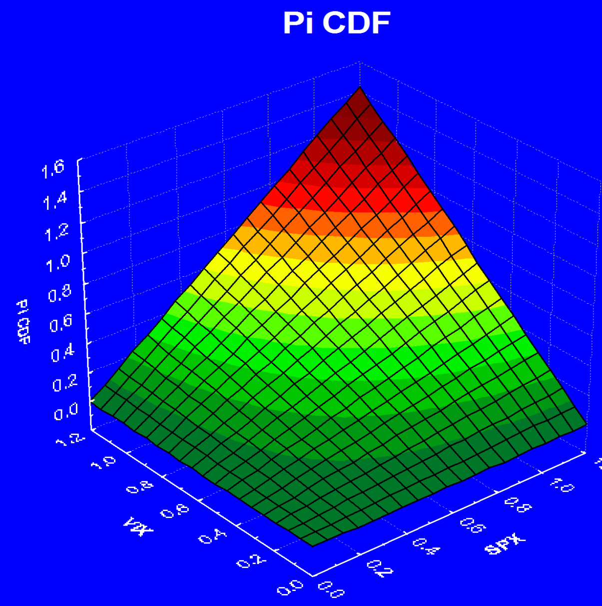


VIX/SPX Empirical CDF

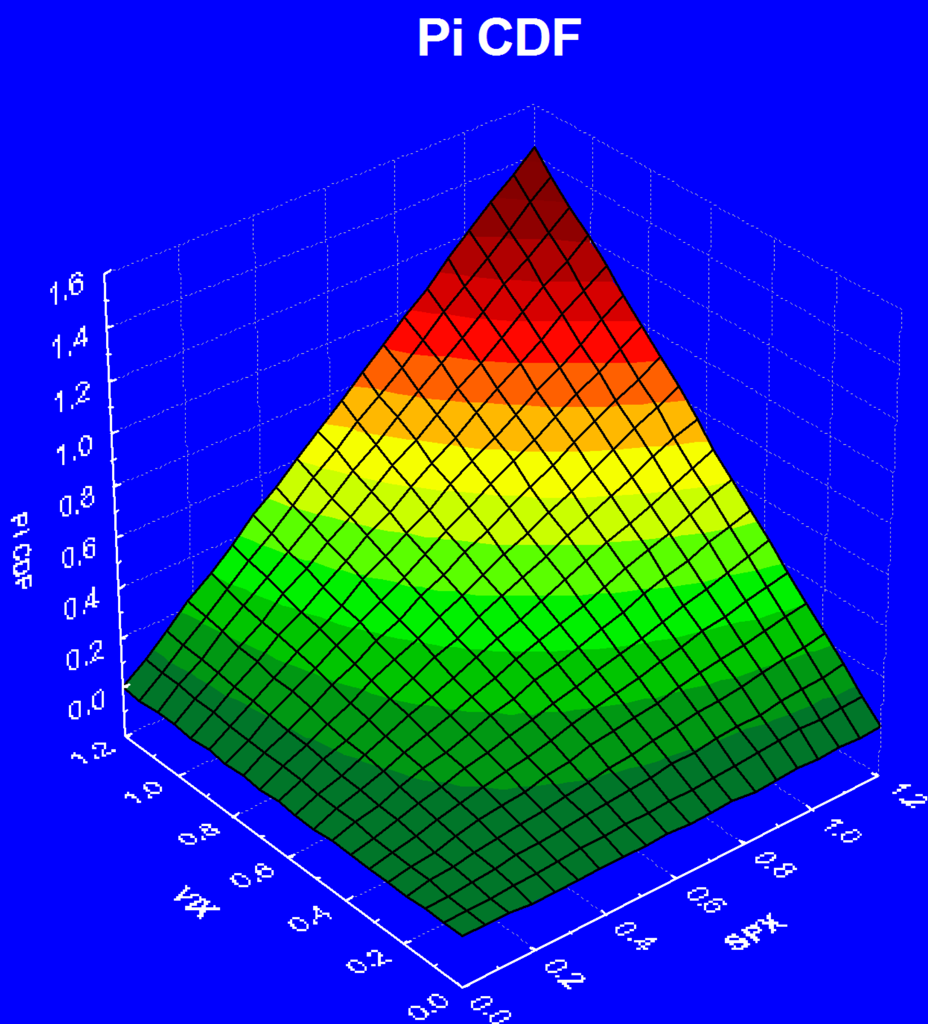


Empirical CDF vs. Π



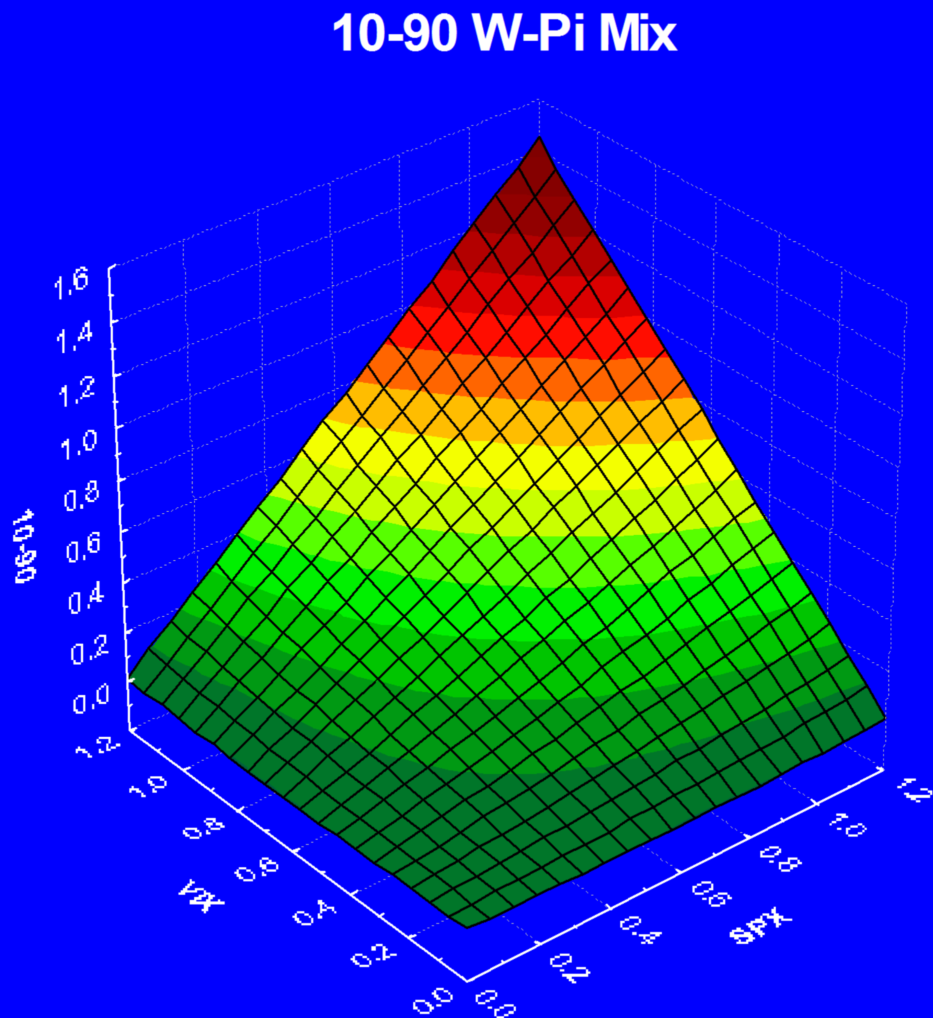


Π CDF

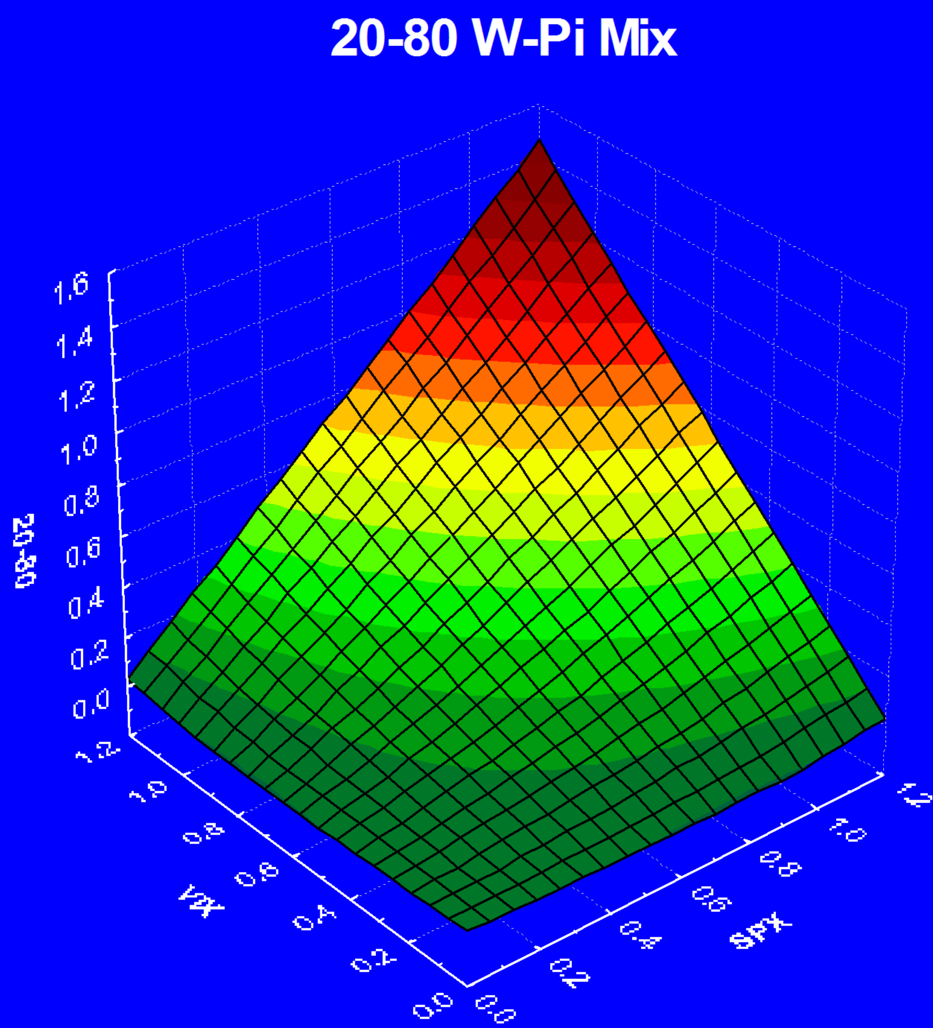


Pi CDF
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
Pi CDF
1.2
1.0
0.8
0.6
0.4
0.2
0.0
VIX
0.0
0.2
0.4
0.6
0.8
1.0
1.2
SPX

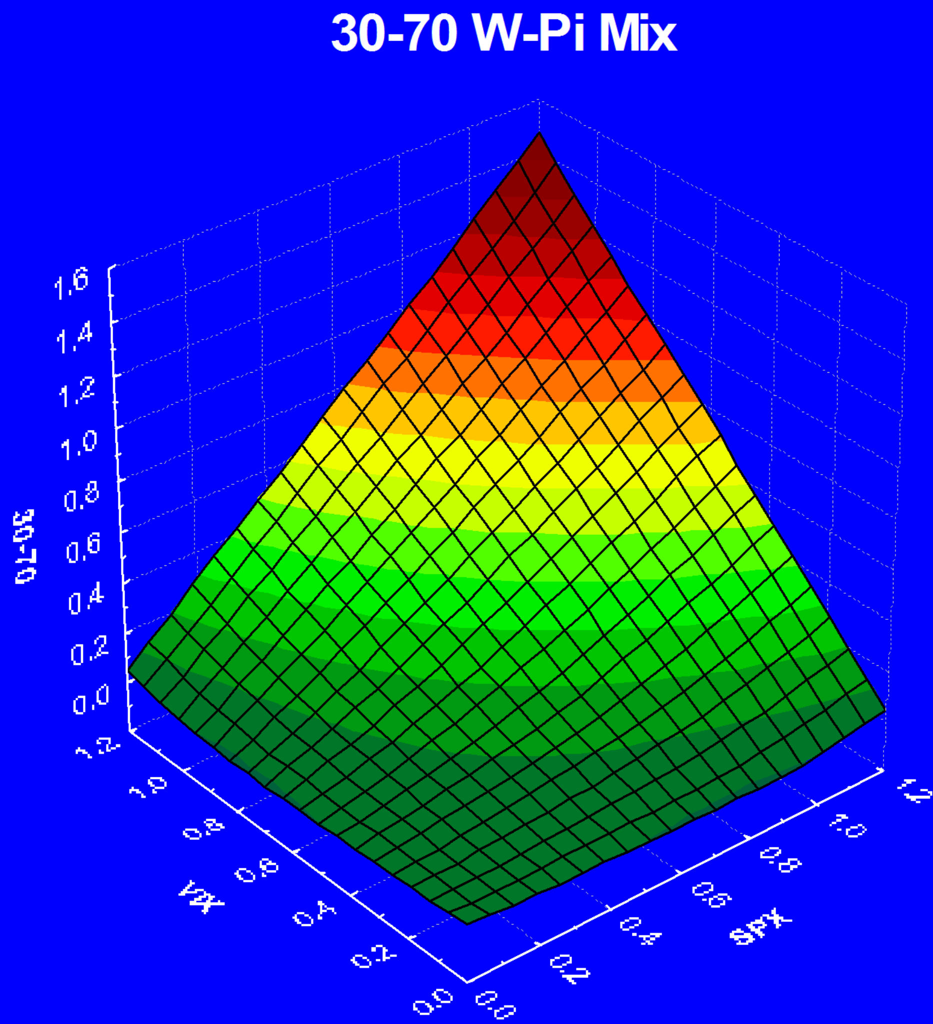
10% W + 90% Π



20% W + 80% Π

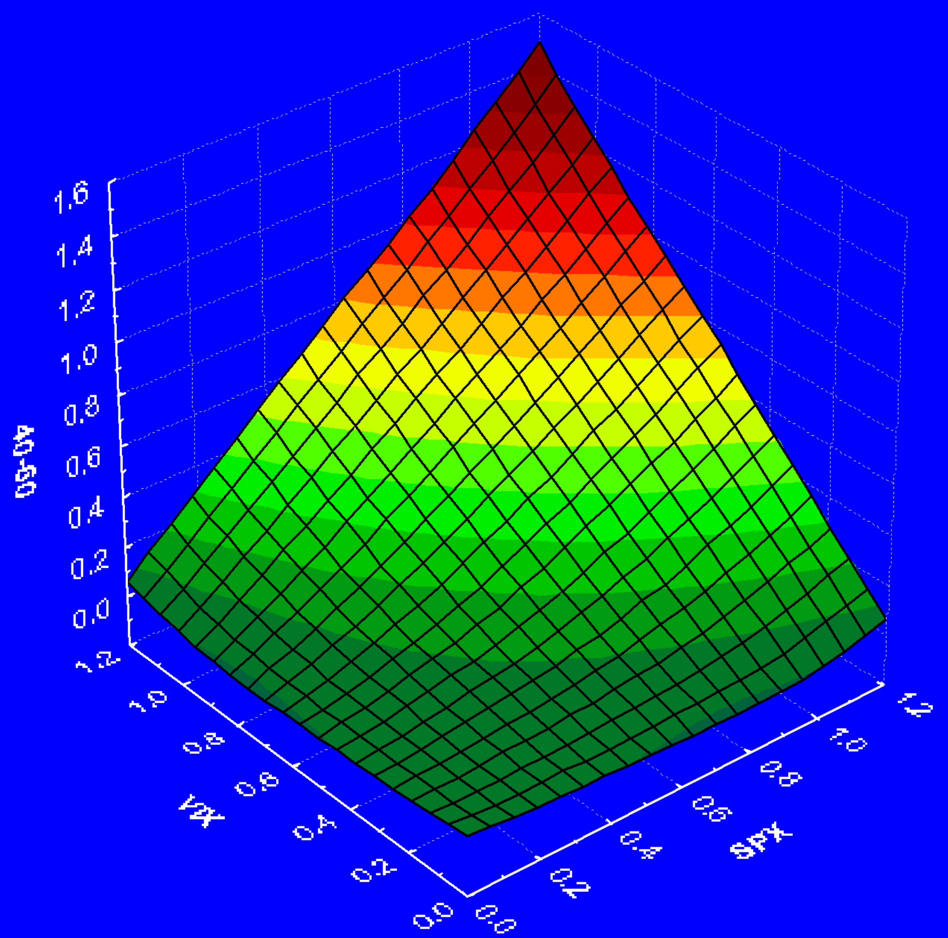


30% W + 70% П

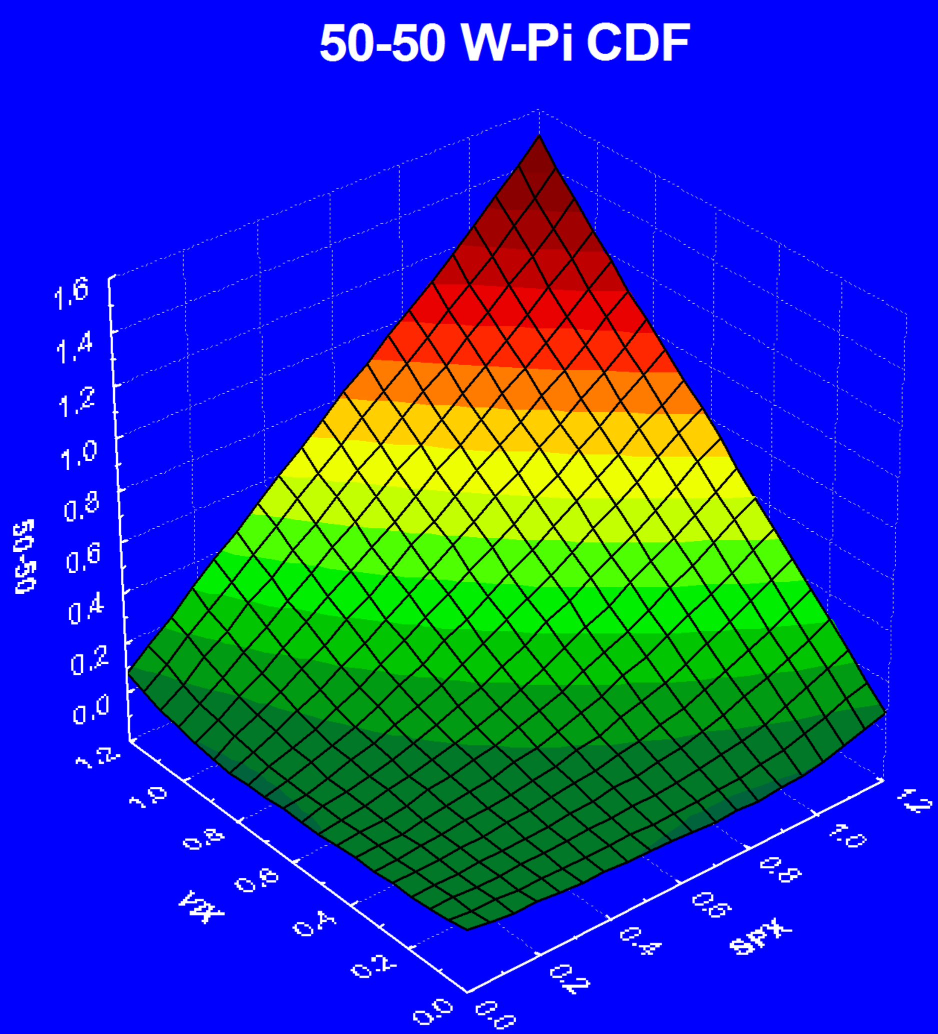

30-70 W-Pi Mix
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
30-70
1.2
1.0
0.8
0.6
0.4
0.2
0.0
VIX
0.0
0.2
0.4
0.6
0.8
1.0
1.2
SPX

40% W + 60% Π

40-60 W-Pi Mix

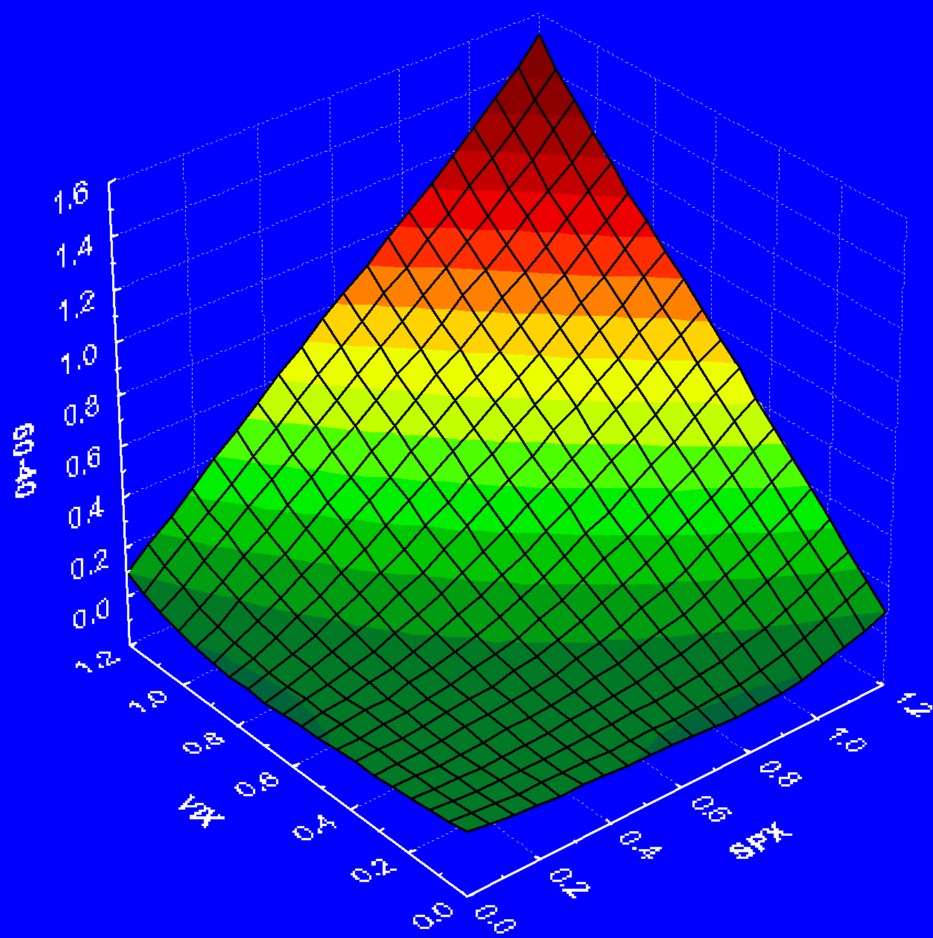


50% W + 50% Π

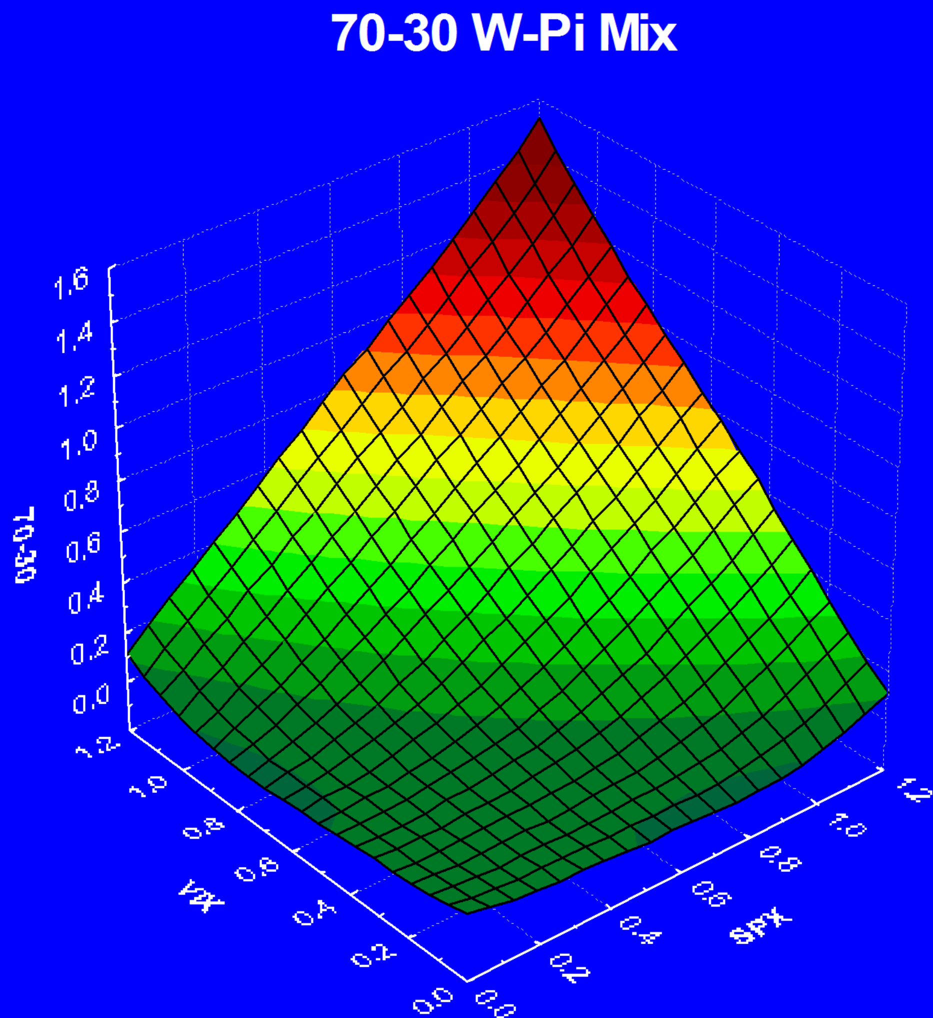


60% W + 40% Π

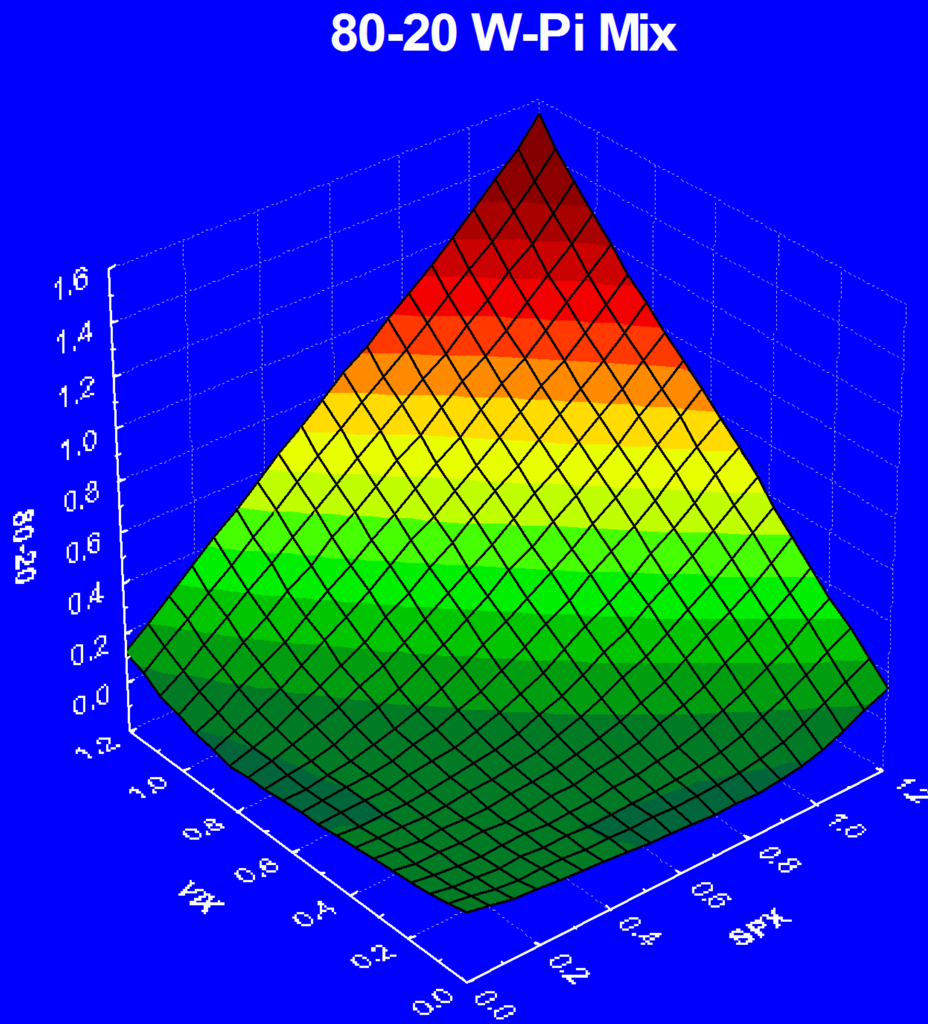
60-40 W-Pi Mix



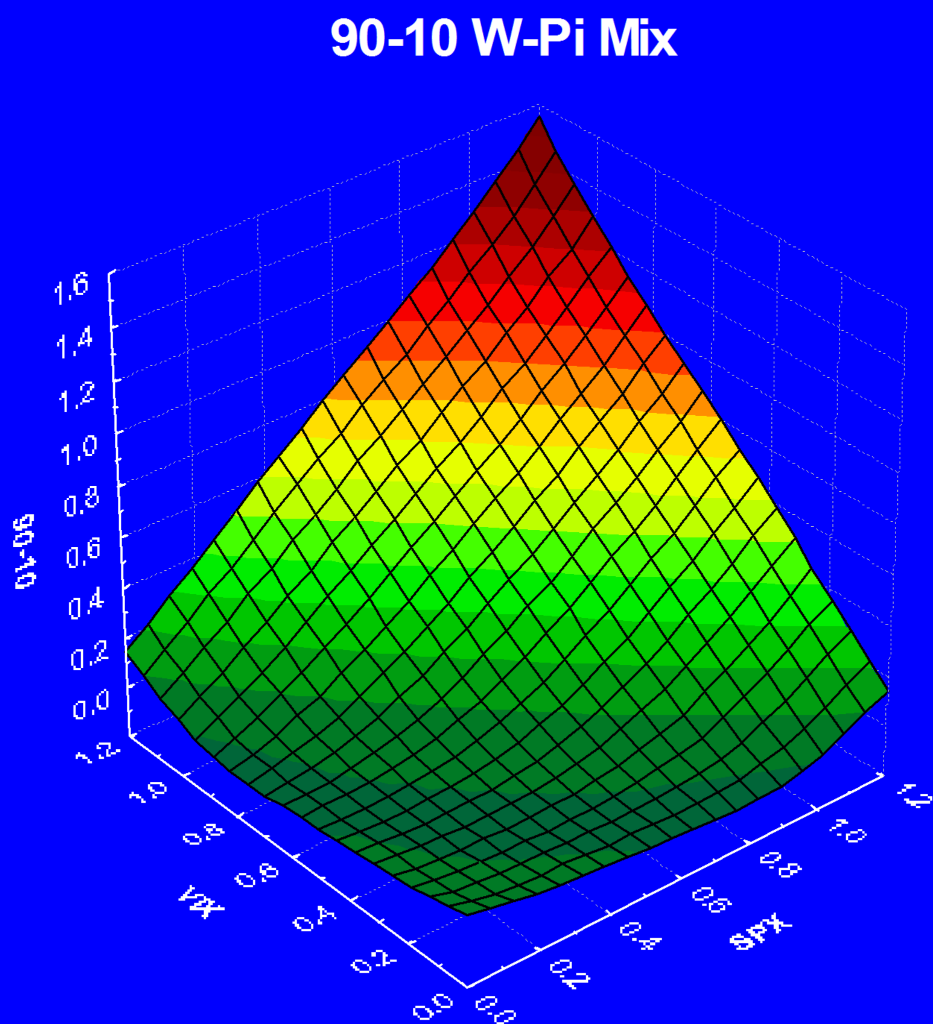
70% W + 30% Π



80% W + 20% П

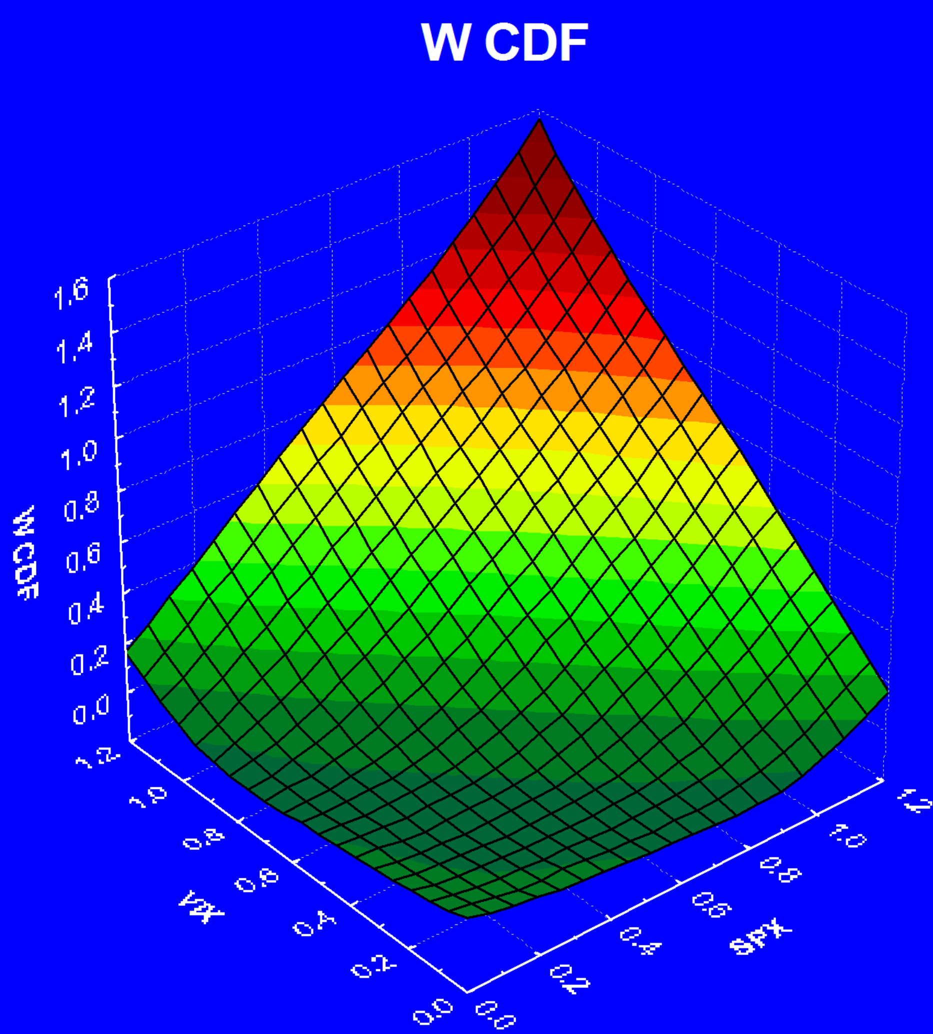

80-20 W-Pi Mix
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
80-20
VIX
SPX
1.2
1.0
0.8
0.6
0.4
0.2
0.0

90% W + 10% Π

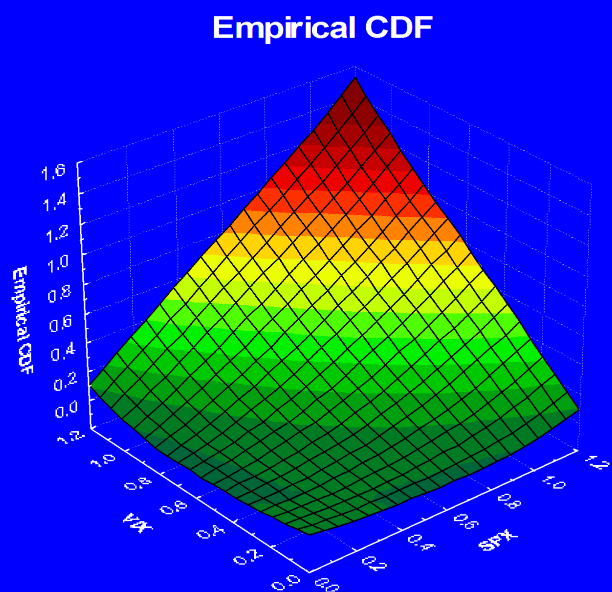


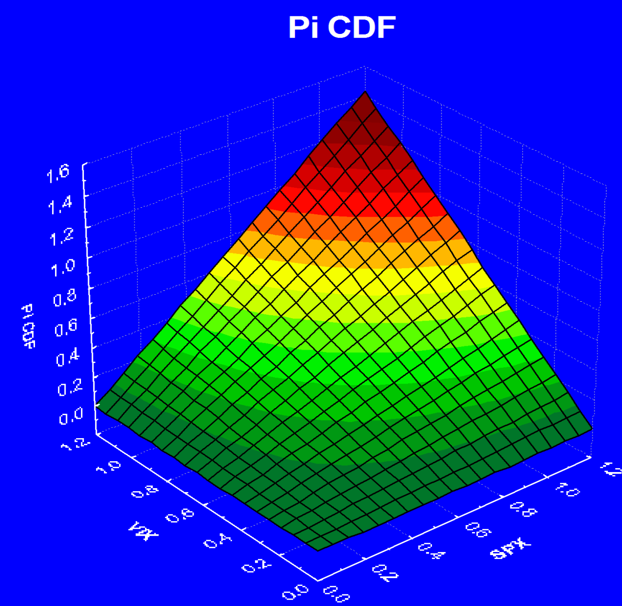
90-10 W-Pi Mix
90-10
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
WX
1.0
0.8
0.6
0.4
0.2
0.0
SPX
0.0
0.2
0.4
0.6
0.8
1.0
1.2

W CDF


W CDF
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
W CDF
VIX
0.0
0.2
0.4
0.6
0.8
1.0
1.2
SPX
0.0
0.2
0.4
0.6
0.8
1.0
1.2

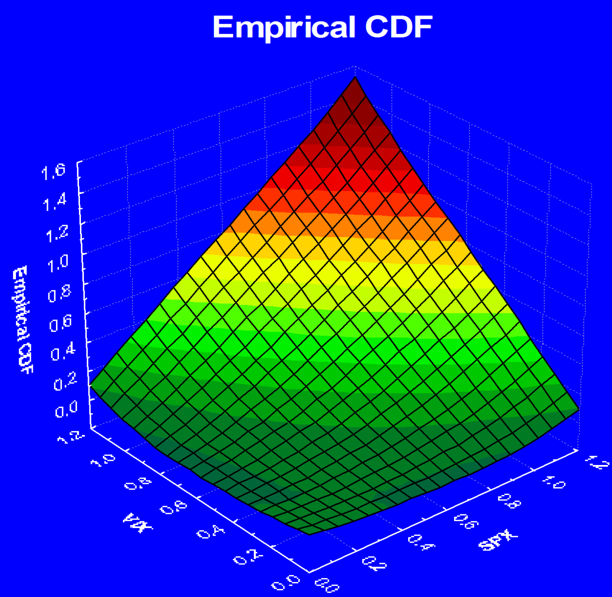
Empirical CDF vs. Π CDF

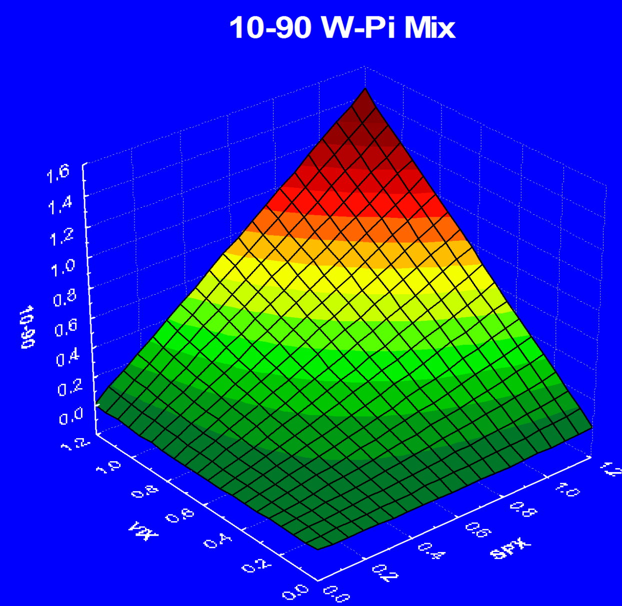




p*	QD*	p**	QD**
0.099	-0.801	0	-1

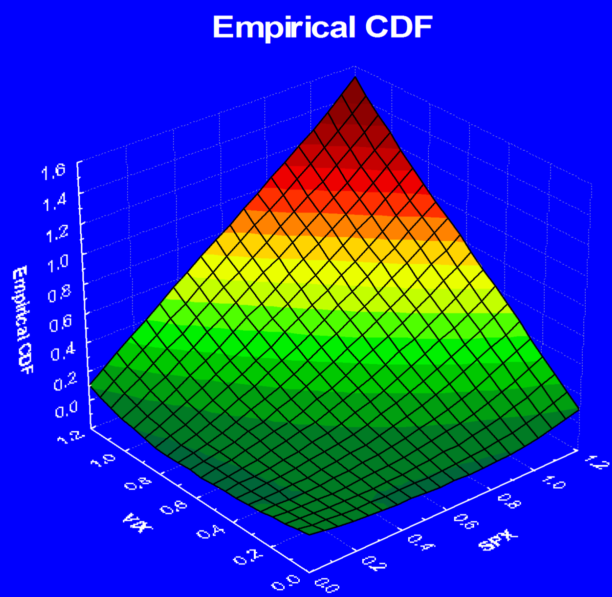
Empirical CDF vs. 10-90 W-Π Mix

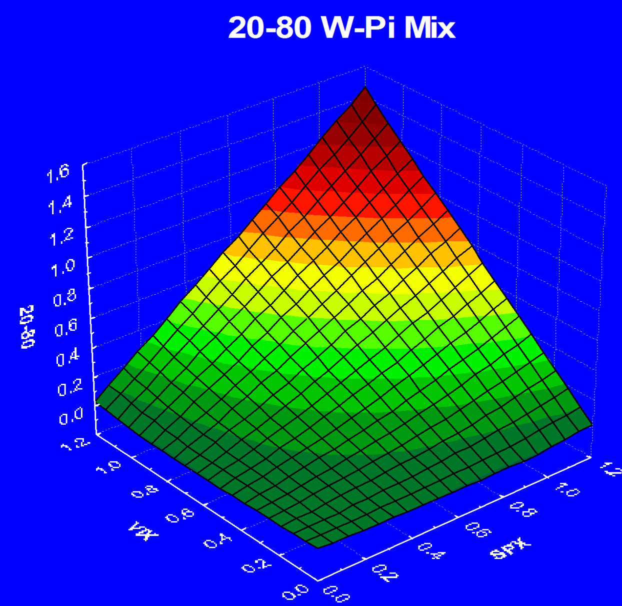




p*	QD*	p**	QD**
0.099	-0.801	0	-1

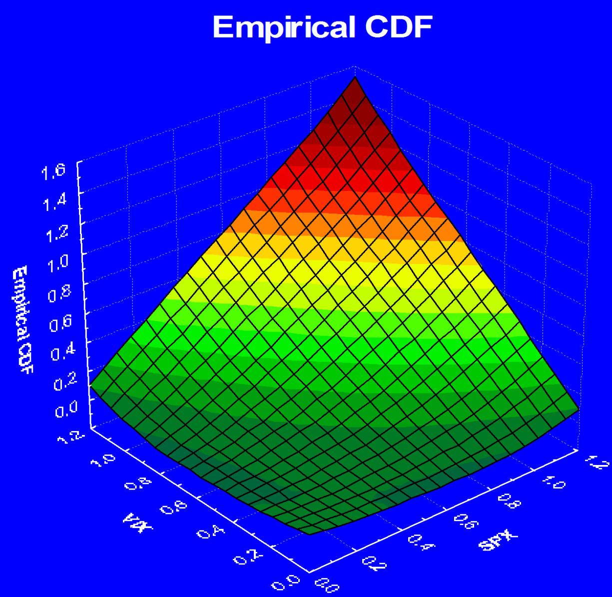
Empirical CDF vs. 20-80 W-Π Mix

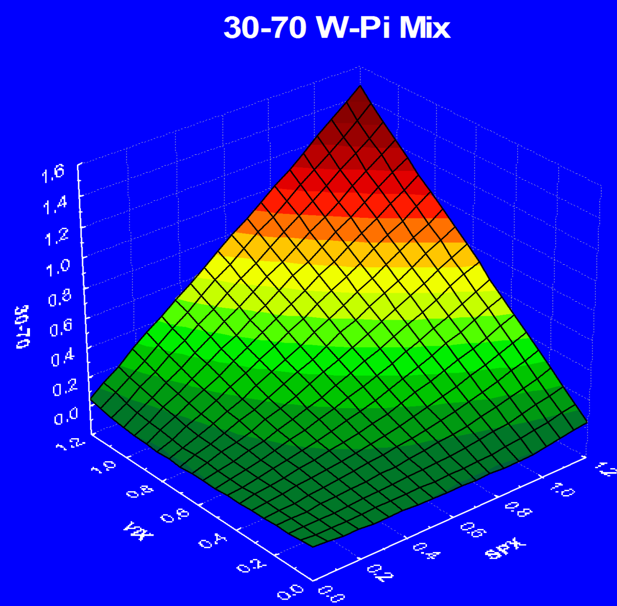




p*	QD*	p**	QD**
0.099	-0.801	0	-1

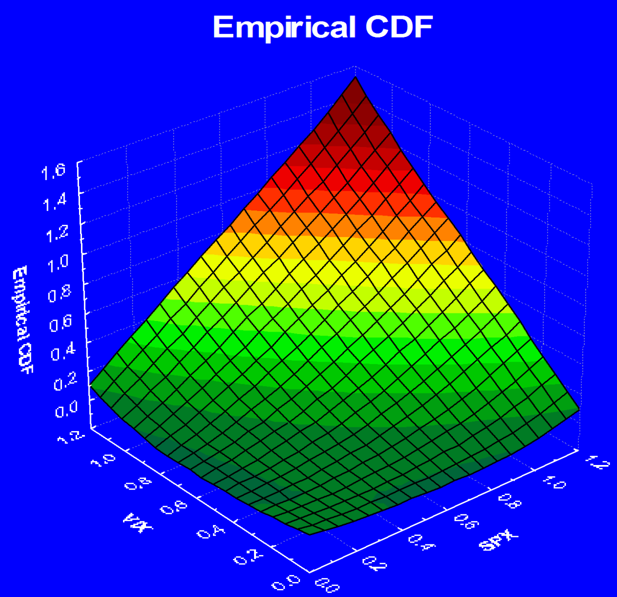
Empirical CDF vs. 30-70 W-Π Mix

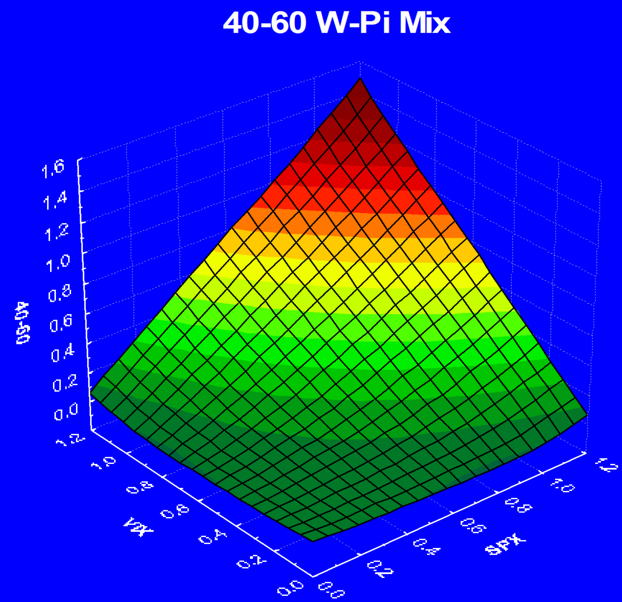




p*	QD*	p**	QD**
0.099	-0.801	0	-1

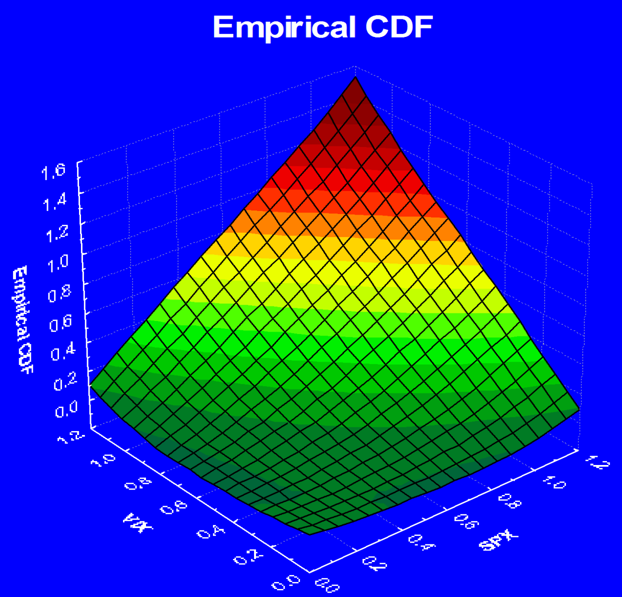
Empirical CDF vs. 40-60 W-Π Mix

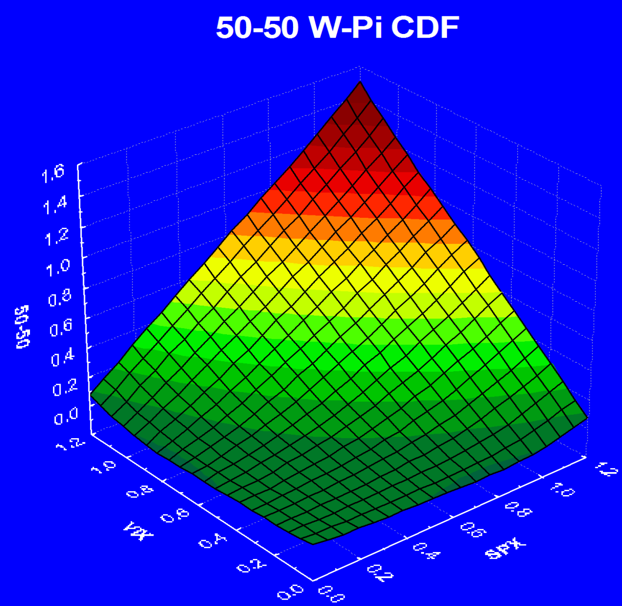




p*	QD*	p**	QD**
0.119	-0.761	0.025	-0.950

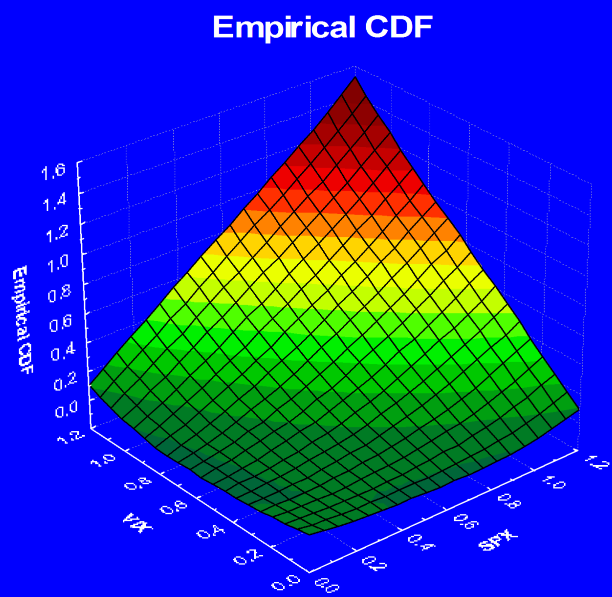
Empirical CDF vs. 50-50 W-Π Mix

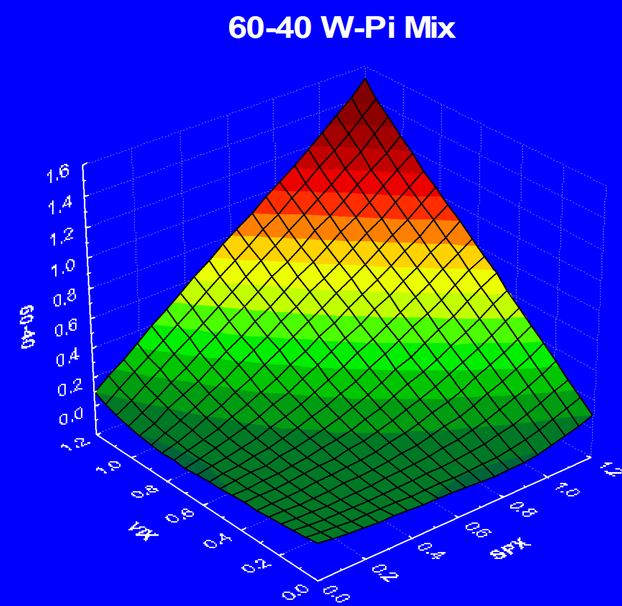




p*	QD*	p**	QD**
0.236	-0.527	0.171	-0.658

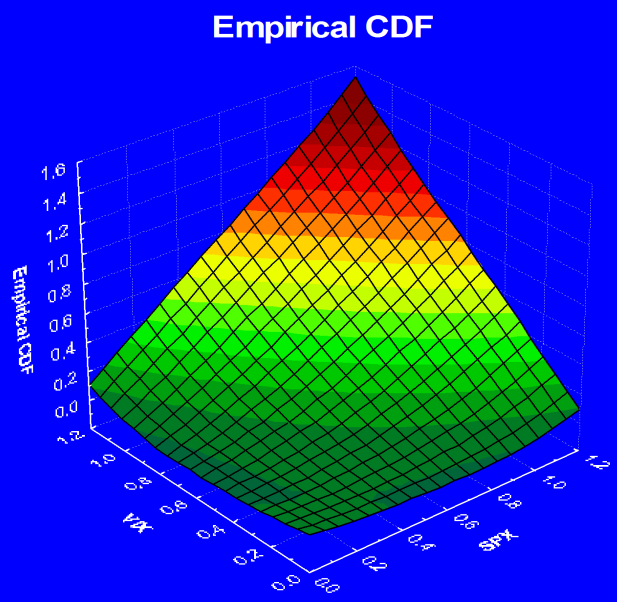
Empirical CDF vs. 60-40 W-Π Mix

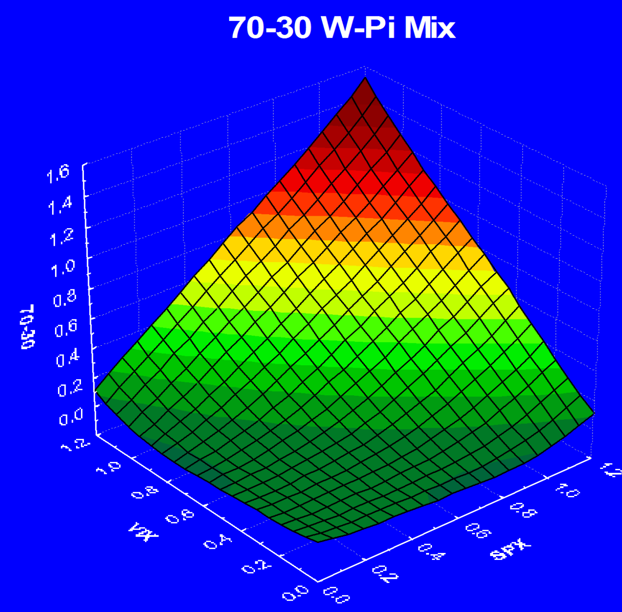




p*	QD*	p**	QD**
0.414	-0.173	0.392	-0.216

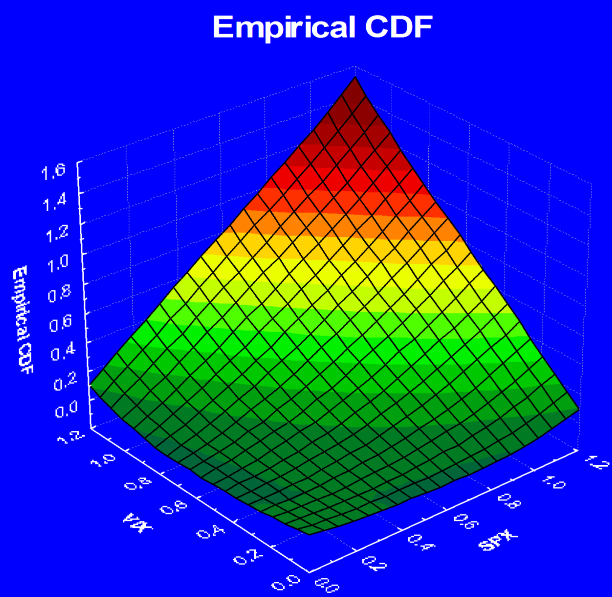
Empirical CDF vs. 70-30 W-Π Mix

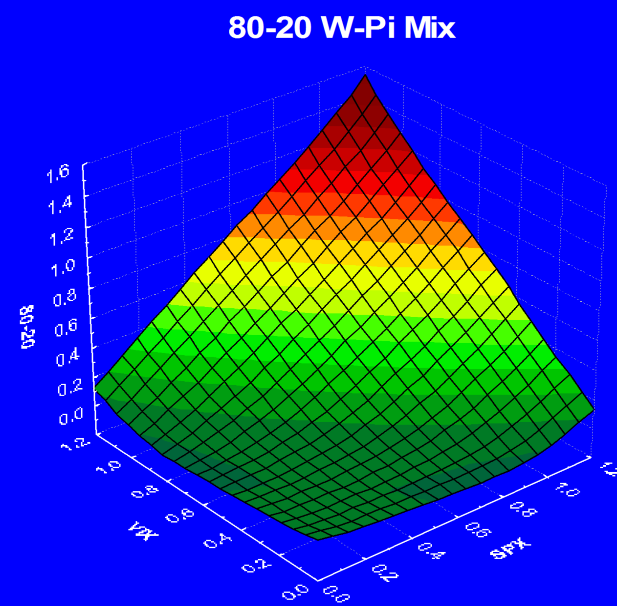




p*	QD*	p**	QD**
0.639	0.278	0.673	0.347

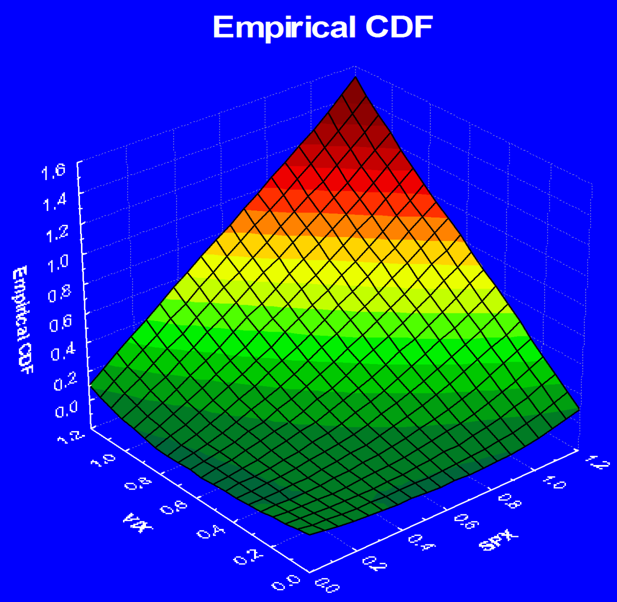
Empirical CDF vs. 80-20 W-Π Mix

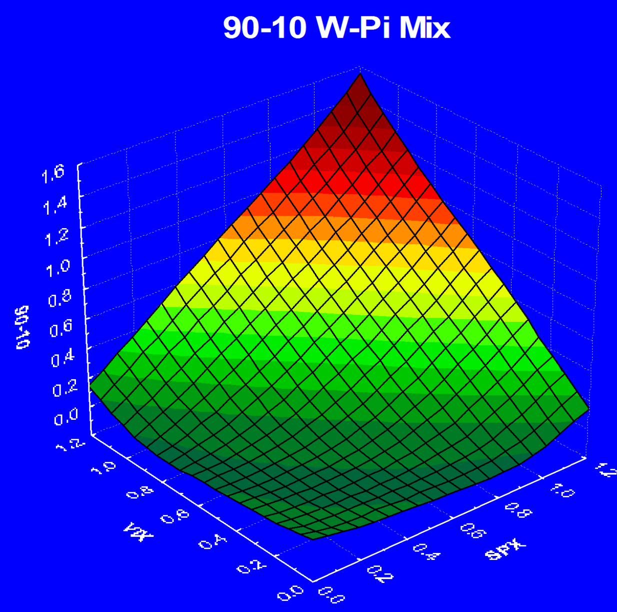




p*	QD*	p**	QD**
0.801	0.601	0.875	0.750

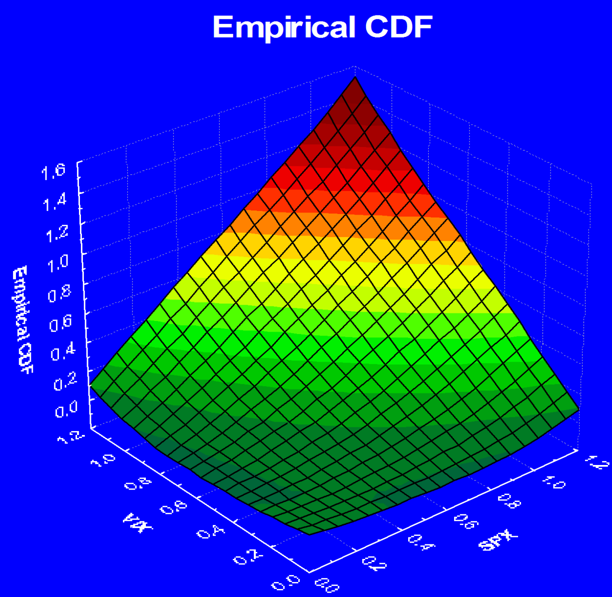
Empirical CDF vs. 90-10 W-Π Mix

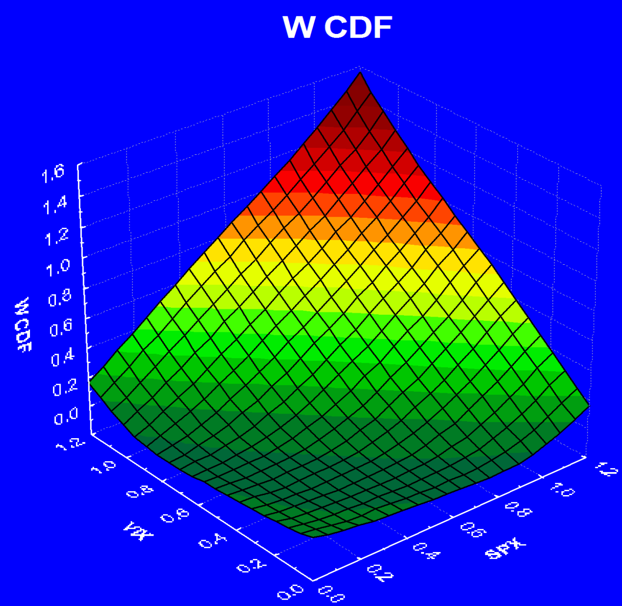




p*	QD*	p**	QD**
0.896	0.792	0.994	0.989

Empirical CDF vs. W CDF





p*	QD*	p**	QD**
0.901	0.801	1	1

VIX/SPX Summary

%W	%Π	p*	QD*	p**	QD**
0	100	0.099	-0.801	0	-1
10	90	0.099	-0.801	0	-1
20	80	0.099	-0.801	0	-1
30	70	0.099	-0.801	0	-1
40	60	0.119	-0.761	0.025	-0.950
50	50	0.236	-0.527	0.171	-0.658
60	40	0.414	-0.173	0.392	-0.216
64	36	0.501	0.001	0.501	0.001
70	30	0.639	0.278	0.673	0.347
80	20	0.801	0.601	0.875	0.750
90	10	0.896	0.792	0.994	0.989
100	0	0.901	0.801	1	1

Partial RTI and SI

Let μ be a probability measure for Y.

The μ-degree of RTI of Y in X is

$\mu\{y : \mathsf{P}[Y > y|X > x]$ is a nondecreasing function of $x\}$

The μ-degree of SI of Y in X is

$\mu\{y : \mathsf{P}[Y > y|X = x]$ is a nondecreasing function of $x\}$

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