A Skewed Look at Bivariate and Multivariate Order Statistics

Prof. N. Balakrishnan

Dept. of Mathematics & Statistics

McMaster University, Canada

bala@mcmaster.ca

Presented with great pleasure as

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Plenary Lecture

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at Tartu Conference, 2007

Roadmap

1. Order Statistics
2. Skew-Normal Distribution
3. OS from BVN Distribution
4. Generalized Skew-Normal Distribution
5. OS from TVN Distribution
6. OS Induced by Linear Functions
7. OS from BV and TV t Distributions
8. Bibliography

Order Statistics

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- If we arrange these X_i's in increasing order of magnitude, we obtain the so-called *order statistics*, denoted by

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n},$$

which are clearly dependent.

Order Statistics (cont.)

- Using multinomial argument, we readily have for $r = 1, \cdots, n$

$$
\begin{aligned}
&\Pr\left(x < X_{r:n} \le x + \delta x\right) \\
&= \frac{n!}{(r-1)!(n-r)!} \left\{F(x)\right\}^{r-1} \left\{F(x+\delta x) - F(x)\right\} \\
&\quad \times \left\{1 - F(x+\delta x)\right\}^{n-r} + O\left((\delta x)^2\right).
\end{aligned}
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Order Statistics (cont.)

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- From this, we obtain the pdf of $X_{r:n}$ as (for $x \in \boldsymbol{R}$)

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} \left\{F(x)\right\}^{r-1} \left\{1 - F(x)\right\}^{n-r} f(x).$$

Order Statistics (cont.)

- Similarly, we obtain the joint pdf of $(X_{r:n}, X_{s:n})$ as (for $1 \le r < s \le n$ and $x < y$)

$$\begin{aligned} f_{r,s:n}(x,y) &= \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \{F(x)\}^{r-1} f(x) \\ &\times \{F(y) - F(x)\}^{s-r-1} \{1 - F(y)\}^{n-s} f(y). \end{aligned}$$

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- From the pdf and joint pdf, we can derive, for example, means, variances and covariances of order statistics, and also study their dependence structure.
- The area of order statistics has a long and rich history, and a very vast literature.

Order Statistics (cont.)

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  H.A. David (1970, 1981)

  B. Arnold & N. Balakrishnan (1989)

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  B. Arnold, N. Balakrishnan & H.N. Nagaraja (1992)

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- However, most of the literature on order statistics have focused on the independent case, and very little on the dependent case.

Skew-normal Distribution

- The *skew-normal distribution* has pdf

$$\varphi(x) = 2\,\Phi(\lambda x)\phi(x), \quad x \in R, \quad \lambda \in R,$$

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where $\phi(\cdot)$ and $\Phi(\cdot)$ are standard normal pdf and cdf, respectively.

- Note that

$\lambda \in R$ is a shape parameter;

$\lambda = 0$ corresponds to std. normal case;

$\lambda \to \infty$ corresponds to half normal case;

Location and scale parameters can be introduced into the model as well.

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  Birnbaum (1950)

  Nelson (1964)

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  O'Hagan & Leonard (1976)
- Interpretation through hidden truncation / selective reporting is due to Arnold & Beaver (2002, *Test*) for univariate/multivariate case.

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$$L_n(\lambda) = \int_{-\infty}^{\infty} \{\Phi(\lambda x)\}^n \; \phi(x) \, dx.$$

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- Clearly, $L_0(\lambda) = 1 \ \forall \lambda \in R$.
- Furthermore,

$$\int_{-\infty}^{\infty} \left\{\Phi(\lambda x) - \frac{1}{2}\right\}^{2n+1} \ \phi(x)\, dx = 0$$

since the integrand is an odd function of x, we obtain:

Skew-Normal Distribution (cont.)

$$L_{2n+1}(\lambda) = \sum_{i=1}^{2n+1} (-1)^{i+1} \frac{1}{2^i} \binom{2n+1}{i} L_{2n+1-i}(\lambda), \quad \lambda \in R,$$

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for $n = 0, 1, 2, \ldots$

- For $n = 0$, we simply obtain

$$L_1(\lambda) = \int_{-\infty}^{\infty} \Phi(\lambda x)\, \phi(x)\, dx = \frac{1}{2} L_0(\lambda) = \frac{1}{2}, \quad \lambda \in R,$$

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which leads to the skew-normal density

$$\varphi_1(x; \lambda) = 2\, \Phi(\lambda x) \phi(x), \quad x \in R, \quad \lambda \in R.$$

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$$\begin{aligned} \frac{dL_2(\lambda)}{d\lambda} &= 2\int_{-\infty}^{\infty} x\ \Phi(\lambda x)\ \phi(\lambda x)\ \phi(x)\ dx \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \Phi(\lambda x)\ x\ \exp\left\{-\frac{1}{2}x^2(1+\lambda^2)\right\}\ dx \\ &= \frac{\lambda}{\pi(1+\lambda^2)\sqrt{1+2\lambda^2}}. \end{aligned}$$

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- Solving this differential equation, we obtain

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Remark 1: Interestingly, this family also includes standard normal (when $\lambda = 0$) and the half normal (when $\lambda \to \infty$) distributions, just as $\varphi_1(x;\lambda)$ does.

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Remark 3: Evidently, in the special case when $\lambda = 1$, the densities $\varphi_1(x;\lambda), \varphi(x;\lambda_2)$ and $\varphi_3(x;\lambda)$ become the densities of the largest OS in samples of size 2, 3 and 4, respectively, from $N(0,1)$ distribution.

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Remark 4: In addition, the integral $L_n(\lambda)$ is also involved in the means of OS from $N(0,1)$ distribution. For example, with $\mu_{m:m}$ denoting the mean of the largest OS in a sample of size m from $N(0,1)$, we have

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$$\mu_{2:2} = \frac{L_0}{\sqrt{\pi}}, \;\; \mu_{3:3} = \frac{3L_1(1)}{\sqrt{\pi}}, \;\; \mu_{4:4} = \frac{6L_2(1)}{\sqrt{\pi}}, \;\; \mu_{5:5} = \frac{10L_3(1)}{\sqrt{\pi}}.$$

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- Next, can we have a *similar skewed* look at OS from BV and MV normal distributions?
- How about other distributions?

OS from BVN Distribution

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  S.S. Gupta and K.C.S. Pillai (1965)

  A.P. Basu and J.K. Ghosh (1978)

  H.N. Nagaraja (1982)

  N. Balakrishnan (1993)

  M. Cain (1994)

  M. Cain and E. Pan (1995)

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where $X \sim N(0, 1)$ independently of $(Y_1, Y_2) \sim BVN(0, 0, 1, 1, \rho)$.

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- It should be mentioned that $Z_{\lambda_1,\lambda_2,\rho}$ belongs to

  $SUN_{1,2}(0,0,1,\Omega^*)$ [Arellano-Valle & Azzalini (2006)]

  $CSN_{1,2}$ [Farias, Molina & A.K. Gupta (2004)]

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  with $\lambda_1, \lambda_2 \in R$, $|\rho| < 1$, and $\Phi(\cdot,\cdot;\rho)$ denoting the cdf of $BVN(0,0,1,1,\rho)$.

- For determining $c(\lambda_1,\lambda_2,\rho)$, we note that

$$c(\lambda_1,\lambda_2,\rho) \equiv \frac{1}{a(\lambda_1,\lambda_2,\rho)} = \frac{1}{P\left(Y_1 < \lambda_1 X, Y_2 < \lambda_2 X\right)},$$

  where $X \sim N(0,1)$ independently of $(Y_1, Y_2) \sim BVN(0,0,1,1,\rho)$.

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$$\begin{aligned} a(\lambda_1, \lambda_2, \rho) &= P\left(Y_1 < \lambda_1 X, Y_2 < \lambda_2 X\right) \\ &= \frac{1}{2\pi} \cos^{-1}\left(\frac{-(\rho + \lambda_1\lambda_2)}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right); \end{aligned}$$

[Kotz, Balakrishnan and Johnson (2000)].

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- The generalized skew-normal pdf is then

$$
\varphi(z; \lambda_1, \lambda_2, \rho) = \frac{2\pi}{\cos^{-1}\left(\frac{-(\rho+\lambda_1\lambda_2)}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}} \right)} \phi(z) \Phi(\lambda_1 z, \lambda_2 z; \rho)
$$

for $z, \lambda_1, \lambda_2 \in R,\ |\rho| < 1$.

Generalized Skew-Normal Distribution (cont.)

- Let $\Phi(\cdot, \cdot; \delta)$ denote cdf of $BVN(0, 0, 1, 1, \delta)$, and $\Phi(\cdot; \theta)$ denote cdf of $SN(\theta)$.

Generalized Skew-Normal Distribution (cont.)

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Generalized Skew-Normal Distribution (cont.)

- Let $\Phi(\cdot,\cdot;\delta)$ denote cdf of $BVN(0,0,1,1,\delta)$, and $\Phi(\cdot;\theta)$ denote cdf of $SN(\theta)$.

<u>**Lemma 2**</u>: We then have

$$\begin{aligned}
\Phi(0,0;\delta) &= \frac{1}{2\pi}\cos^{-1}(-\delta), \\
\Phi(\gamma x,0;\delta) &= \Phi(0,\gamma x;\delta) = \frac{1}{2}\Phi\left(\gamma x;\frac{-\delta}{\sqrt{1-\delta^2}}\right), \\
\Phi(\gamma_1 x,\gamma_2 x;\delta) &= \frac{1}{2}\left\{\Phi(\gamma_1 x;\eta_1)+\Phi(\gamma_2 x;\eta_2)-I(\gamma_1\gamma_2)\right\},
\end{aligned}$$

where $I(a)=0$ if $a>0$ and 1 if $a<0$,

$$\eta_1 = \frac{1}{\sqrt{1-\delta^2}}\left(\frac{\gamma_2}{\gamma_1}-\delta\right), \quad \eta_2 = \frac{1}{\sqrt{1-\delta^2}}\left(\frac{\gamma_1}{\gamma_2}-\delta\right).$$

Generalized Skew-Normal Distribution (cont.)

Theorem 1: If $M(t; \lambda_1, \lambda_2, \rho)$ is the MGF of $Z_{\lambda_1,\lambda_2,\rho} \sim GSN(\lambda_1, \lambda_2, \rho)$, then

Generalized Skew-Normal Distribution (cont.)

Theorem 1: If $M(t;\lambda_1,\lambda_2,\rho)$ is the MGF of $Z_{\lambda_1,\lambda_2,\rho} \sim GSN(\lambda_1,\lambda_2,\rho)$, then

$$\begin{aligned} M(t;\lambda_1,\lambda_2,\rho) &= \frac{2\pi}{\cos^{-1}\left(\frac{-(\rho+\lambda_1\lambda_2)}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right)} e^{t^2/2} \\ &\times \Phi\left(\frac{\lambda_1 t}{\sqrt{1+\lambda_1^2}}, \frac{\lambda_2 t}{\sqrt{1+\lambda_2^2}}; \frac{\rho+\lambda_1\lambda_2}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right). \end{aligned}$$

Generalized Skew-Normal Distribution (cont.)

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Corollary 1: Theorem 1 yields, for example,

$$E\left[Z_{\lambda_1,\lambda_2,\rho}\right] = \frac{\sqrt{\pi/2}}{\cos^{-1}\left(\frac{-(\rho+\lambda_1\lambda_2)}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right)} \left\{\frac{\lambda_1}{\sqrt{1+\lambda_1^2}} + \frac{\lambda_2}{\sqrt{1+\lambda_2^2}}\right\}.$$

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- Let $W_{1:3} = \min(W_1, W_2, W_3) < W_{2:3} < W_{3:3} = \max(W_1, W_2, W_3)$ denote the order statistics from (W_1, W_2, W_3).

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- Let $F_i(t; \Sigma)$ denote the cdf of $W_{i:3}, i = 1, 2, 3$.

OS from TVN Distribution (cont.)

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$$F_3(t;\Sigma) = a(\boldsymbol{\theta}_1)\Phi\left(\frac{t}{\sigma_1};\boldsymbol{\theta}_1\right) + a(\boldsymbol{\theta}_2)\Phi\left(\frac{t}{\sigma_2};\boldsymbol{\theta}_2\right) + a(\boldsymbol{\theta}_3)\Phi\left(\frac{t}{\sigma_3};\boldsymbol{\theta}_3\right),$$

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where $\Phi(\cdot;\boldsymbol{\theta})$ denotes the cdf of $GSN(\boldsymbol{\theta})$,

$$a(\lambda_1,\lambda_2,\rho) = \frac{1}{2\pi}\cos^{-1}\left(\frac{-(\rho+\lambda_1\lambda_2)}{\sqrt{1+\lambda_1^2}\sqrt{1+\lambda_2^2}}\right),$$

$$\begin{aligned}
\boldsymbol{\theta}_1 &= \left(\frac{\frac{\sigma_1}{\sigma_2}-\rho_{12}}{\sqrt{1-\rho_{12}^2}}, \frac{\frac{\sigma_1}{\sigma_3}-\rho_{13}}{\sqrt{1-\rho_{13}^2}}, \frac{\rho_{23}-\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{13}^2}}\right),\\
\boldsymbol{\theta}_2 &= \left(\frac{\frac{\sigma_2}{\sigma_1}-\rho_{12}}{\sqrt{1-\rho_{12}^2}}, \frac{\frac{\sigma_2}{\sigma_3}-\rho_{23}}{\sqrt{1-\rho_{23}^2}}, \frac{\rho_{13}-\rho_{12}\rho_{23}}{\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{23}^2}}\right),\\
\boldsymbol{\theta}_3 &= \left(\frac{\frac{\sigma_3}{\sigma_1}-\rho_{13}}{\sqrt{1-\rho_{13}^2}}, \frac{\frac{\sigma_3}{\sigma_2}-\rho_{23}}{\sqrt{1-\rho_{23}^2}}, \frac{\rho_{12}-\rho_{13}\rho_{23}}{\sqrt{1-\rho_{13}^2}\sqrt{1-\rho_{23}^2}}\right).
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OS from TVN Distribution (cont.)

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OS from TVN Distribution (cont.)

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where $\Phi(\cdot;\theta)$ is the cdf of $SN(\theta)$, and

$$\begin{aligned} \alpha_1 &= \frac{1+\rho_{12}-\rho_{13}-\rho_{23}}{\sqrt{A}}, \\ \alpha_2 &= \frac{1+\rho_{13}-\rho_{12}-\rho_{23}}{\sqrt{A}}, \\ \alpha_3 &= \frac{1+\rho_{23}-\rho_{12}-\rho_{13}}{\sqrt{A}}, \\ A &= 6-\left\{(1+\rho_{12})^2+(1+\rho_{13})^2+(1+\rho_{23})^2\right\} \\ &\quad +2\left(\rho_{12}\rho_{13}+\rho_{12}\rho_{23}+\rho_{13}\rho_{23}\right). \end{aligned}$$

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Remark 5: Similar mixture forms can be derived for the cdf of $W_{1:3}$ and $W_{2:3}$, and from them explicit expressions for their mgf, moments, etc.

OS from TVN Distribution (cont.)

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- Can we make use of these results concerning distributions and moments of OS to develop some efficient inferential methods?

OS from TVN Distribution (cont.)

Problems for further study:

- Can we make use of these results concerning distributions and moments of OS to develop some efficient inferential methods?
- How far can these results be generalized to obtain such explicit expressions for the case of OS from MVN?

OS Induced by Linear Functions

- Let $\boldsymbol{X}_i = (X_{1i}, \ldots, X_{pi})^T,\ i = 1, \ldots, n,$ be iid observations from $MVN(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)^T$ and $\Sigma = ((\sigma_{ij}))$.

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- Let $P = \{i_1, \ldots, i_m\}\ (m \geq 1)$ be a partition of $\{1, \ldots, p\}$, and Q its complementary partition.
- Let $\boldsymbol{C} = (c_1, \ldots, c_p)^T$ be a vector of non-zero constants.

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- Let $P = \{i_1, \ldots, i_m\} \, (m \geq 1)$ be a partition of $\{1, \ldots, p\}$, and Q its complementary partition.
- Let $\boldsymbol{C} = (c_1, \ldots, c_p)^T$ be a vector of non-zero constants.
- Further, let us define for $j = 1, \ldots, n$,

$$X_j = \sum_{i \in P} c_i \, X_{ij} = \boldsymbol{C}_P^T \, \boldsymbol{X}_j, \ \ Y_j = \sum_{i \in Q} c_i \, X_{ij} = \boldsymbol{C}_Q^T \, \boldsymbol{X}_j.$$

OS Induced by Linear Functions (cont.)

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$$
\begin{aligned}
\mu_X &= E(X_j) = \boldsymbol{C}_P^T\,\boldsymbol{\mu}, \\
\mu_Y &= E(Y_j) = \boldsymbol{C}_Q^T\,\boldsymbol{\mu}, \\
\sigma_X^2 &= Var(X_j) = \boldsymbol{C}_P^T\,\Sigma\,\boldsymbol{C}_P, \\
\sigma_Y^2 &= Var(Y_j) = \boldsymbol{C}_Q^T\,\Sigma\,\boldsymbol{C}_Q, \\
\sigma_{X,Y} &= Cov(X_j, Y_j) = \boldsymbol{C}_P^T\,\Sigma\,\boldsymbol{C}_Q, \\
\rho &= \frac{\sigma_{X,Y}}{\sigma_X \sigma_Y} = \frac{\boldsymbol{C}_P^T\,\Sigma\,\boldsymbol{C}_Q}{\sqrt{(\boldsymbol{C}_P^T\,\Sigma\,\boldsymbol{C}_P)(\boldsymbol{C}_Q^T\,\Sigma\,\boldsymbol{C}_Q)}}.
\end{aligned}
$$

OS Induced by Linear Functions (cont.)

- Now, let

$$S_j = X_j + Y_j = \boldsymbol{C}_P^T \boldsymbol{X}_j + \boldsymbol{C}_Q^T \boldsymbol{X}_j = \boldsymbol{C}^T \boldsymbol{X}_j$$

for $j = 1, \ldots, n.$

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- Let $S_{1:n} \leq S_{2:n} \leq \cdots \leq S_{n:n}$ denote the order statistics of S_j's.

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  for $j = 1, \ldots, n.$

- Let $S_{1:n} \leq S_{2:n} \leq \cdots \leq S_{n:n}$ denote the order statistics of S_j's.

- Let $\boldsymbol{X}_{[k:n]}$ be the *k-th induced multivariate order statistic*; i.e.,

$$\boldsymbol{X}_{[k:n]} = \boldsymbol{X}_j \quad \text{whenever} \quad S_{k:n} = S_j.$$

OS Induced by Linear Functions (cont.)

Example 1: While analyzing extreme lake levels in hydrology, the annual maximum level at a location in the lake is a combination of the daily water level averaged over the entire lake and the up surge in local water levels due to wind effects at that site [Song, Buchberger & Deddens (1992)].

OS Induced by Linear Functions (cont.)

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Example 2: While evaluating the performance of students in a course, the final grade may often be a weighted average of the scores in mid-term tests and the final examination.

OS Induced by Linear Functions (cont.)

Theorem 4: We have

$$X_{[k:n]} = \boldsymbol{C}_P^T\,\boldsymbol{X}_{[k:n]} \quad \text{and} \quad Y_{[k:n]} = \boldsymbol{C}_Q^T\,\boldsymbol{X}_{[k:n]}.$$

OS Induced by Linear Functions (cont.)

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Consequently, we readily obtain

$$E(X_{[k:n]}) \quad = \quad \boldsymbol{C}_P^T\,\boldsymbol{\mu} + \alpha_{k:n}\left\{\frac{\boldsymbol{C}_P^T\Sigma\boldsymbol{C}_P + \boldsymbol{C}_P^T\Sigma\boldsymbol{C}_Q}{\sqrt{\boldsymbol{C}_P^T\Sigma\boldsymbol{C}_P + \boldsymbol{C}_Q^T\Sigma\boldsymbol{C}_Q + 2\boldsymbol{C}_P^T\Sigma\boldsymbol{C}_Q}}\right\},$$

where $\alpha_{k:n}$ is the mean of the k-th OS from a sample of size n from $N(0,1)$.

OS Induced by Linear Functions (cont.)

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where $\alpha_{k:n}$ is the mean of the k-th OS from a sample of size n from $N(0, 1)$.

Similar expressions can be derived for $Var(X_{[k:n]})$ and other moments.

OS Induced by Linear Functions (cont.)

- By choosing the partitions P and Q suitably, we have the following results for within a concomitant OS.

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Theorem 5: For $k = 1, \ldots, n,$ we obtain

OS Induced by Linear Functions (cont.)

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Theorem 5: For $k = 1, \ldots, n,$ we obtain

$$\begin{aligned} E(X_{i[k:n]}) &= \mu_i + \alpha_{k:n}\left\{\frac{\sum_{r=1}^{p} c_r\sigma_{ir}}{\sqrt{\sum_{s=1}^{p}\sum_{r=1}^{p} c_r c_s \sigma_{rs}}}\right\}, \ i = 1, \ldots, p, \\ Var(X_{i[k:n]}) &= \sigma_{ii} - (1-\beta_{k,k:n})\left\{\frac{\left(\sum_{r=1}^{p} c_r\sigma_{ir}\right)^2}{\sum_{s=1}^{p}\sum_{r=1}^{p} c_r c_s \sigma_{rs}}\right\}, \ i = 1, \ldots, p, \\ Cov(X_{i[k:n]}, X_{j[k:n]})) &= \sigma_{ij} - (1-\beta_{k,k:n})\left\{\frac{\sum_{s=1}^{p}\sum_{r=1}^{p} c_r c_s \sigma_{ir}\sigma_{js}}{\sum_{s=1}^{p}\sum_{r=1}^{p} c_r c_s \sigma_{rs}}\right\}, \\ & \qquad 1 \le i < j \le p, \end{aligned}$$

where $\beta_{k,k:n}$ is the variance of the k-th OS from a sample of size n from $N(0,1)$.

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Theorem 6: For $1 \leq k < \ell < n,$ we obtain

OS Induced by Linear Functions (cont.)

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Theorem 6: For $1 \le k < \ell < n$, we obtain

$$\begin{aligned} Cov(X_{i[k:n]}, X_{i[\ell:n]}) &= \beta_{k,\ell:n} \left\{ \frac{\left(\sum_{r=1}^{p} c_r \sigma_{ir}\right)^2}{\sum_{s=1}^{p} \sum_{r=1}^{p} c_r c_s \sigma_{rs}} \right\}, \\ & \qquad i = 1, \ldots, p, \\ Cov(X_{i[k:n]}, X_{j[\ell:n]})) &= \beta_{k,\ell:n}) \left\{ \frac{\sum_{s=1}^{p} \sum_{r=1}^{p} c_r c_s \sigma_{ir} \sigma_{js}}{\sum_{s=1}^{p} \sum_{r=1}^{p} c_r c_s \sigma_{rs}} \right\}, \\ & \qquad 1 \le i < j \le p, \end{aligned}$$

where $\beta_{k,\ell:n}$ is covariance between k-th and ℓ-th OS from a sample of size n from $N(0,1)$.

OS Induced by Linear Functions (cont.)

Corollary 3: $Var(X_{[k:n]})$ and $Var(Y_{[k:n]})$ can be rewritten as

OS Induced by Linear Functions (cont.)

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$$Var(X_{[k:n]}) = \sigma_X^2 - (1-\beta_{k,k:n})\left\{\frac{\left(a\sigma_X^2 + b\rho\sigma_X\sigma_Y\right)^2}{\Delta}\right\},$$

$$Var(Y_{[k:n]}) = \sigma_Y^2 - (1-\beta_{k,k:n})\left\{\frac{\left(b\sigma_Y^2 + a\rho\sigma_X\sigma_Y\right)^2}{\Delta}\right\},$$

where $\Delta = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y$.

OS Induced by Linear Functions (cont.)

Corollary 3: $Var(X_{[k:n]})$ and $Var(Y_{[k:n]})$ can be rewritten as

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Var(X_{[k:n]}) &= \sigma_X^2 - (1-\beta_{k,k:n})\left\{\frac{\left(a\sigma_X^2 + b\rho\sigma_X\sigma_Y\right)^2}{\Delta}\right\}, \\
Var(Y_{[k:n]}) &= \sigma_Y^2 - (1-\beta_{k,k:n})\left\{\frac{\left(b\sigma_Y^2 + a\rho\sigma_X\sigma_Y\right)^2}{\Delta}\right\},
\end{aligned}
$$

where $\Delta = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho\sigma_X\sigma_Y$.

- Now, since $\beta_{j,k:n} > 0$ [Bickel (1967)] and $\sum_{j=1}^{n} \beta_{j,k:n} = 1$ for $1 \le k \le n$ [Arnold, Balakrishnan & Nagaraja (1992)], we have $0 < \beta_{k,k:n} < 1$.

OS Induced by Linear Functions (cont.)

- Consequently, we observe that

$$Var(X_{[k:n]}) < \sigma_X^2 \quad \text{and} \quad Var(Y_{[k:n]}) < \sigma_Y^2$$

for all $k = 1, \ldots, n$.

OS Induced by Linear Functions (cont.)

- Consequently, we observe that

$$Var(X_{[k:n]}) < \sigma_X^2 \quad \text{and} \quad Var(Y_{[k:n]}) < \sigma_Y^2$$

  for all $k = 1, \ldots, n$.

- Using a similar argument, it can be shown that

$$Var(X_{i[k:n]}) < \sigma_{ii}$$

  for $i = 1, \ldots, p$ and all $k = 1, \ldots, n$.

OS from BV and TV t Distributions

- Arellano-Valle and Azzalini (2006) presented *unified multivariate skew-elliptical distribution* through conditional distributions.

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- Arellano-Valle and Azzalini (2006) presented *unified multivariate skew-elliptical distribution* through conditional distributions.
- A special case of this distribution is a *generalized skew-t_ν distribution*.
- Now, using skew-t_ν and generalized skew-t_ν distributions, distributions and properties of OS from BV and TV t_ν distributions can be studied.

OS from BV and TV t Distributions (cont.)

Problems for further study:

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Problems for further study:

- Can we make use of these results concerning distributions and moments of OS to develop some efficient inferential methods?
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OS from BV and TV t Distributions (cont.)

Problems for further study:

- Can we make use of these results concerning distributions and moments of OS to develop some efficient inferential methods?
- How far can these results be generalized to obtain such explicit expressions for the case of OS from MV t_ν?
- Can we do this work more generally in terms of elliptically contoured distributions, for example?

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