Tõenäosusteooria ja statistika II

Loeng 8

T-jaotus ja *F*-jaotus

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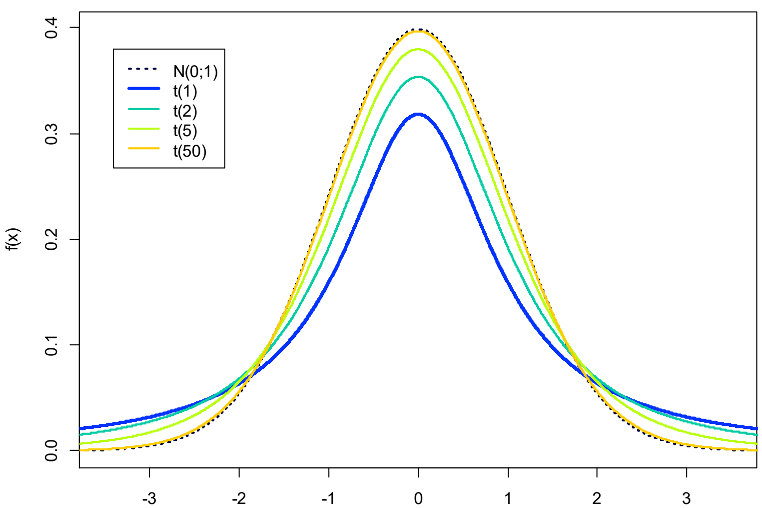
T-jaotuse definitsioon

Juhuslik suurus *X* on *t*-jaotusega vabadusastmete arvuga f, $X \sim t(f)$, kui tema tihedusfunktsioon avaldub kujul

$$f(x) = \frac{\Gamma\left(\frac{f+1}{2}\right)}{\sqrt{f\pi}\ \Gamma\left(\frac{f}{2}\right)} \cdot \left(1 + \frac{x^2}{f}\right)^{-\frac{f+1}{2}}.$$

Vabadusastmete arv *f* on positiivne täisarv ($f \in \mathbb{N}$), x on suvaline reaalarv, $-\infty \le x \le \infty$, $\Gamma(.)$ on gammafunktsioon.

Normaaljaotus ja t-jaotus



Seos normaal- ja hii-ruut jaotusega

Teoreem

Kui juhuslik suurus *X* on normaaljaotusega N(0;1) ja juhuslik suurus *Y* on hii-ruut jaotusega $\chi(f)$, kusjuures *X* ja *Y* on sõltumatud, siis

$$Z = \frac{X}{\sqrt{Y/f}} \sim t(f).$$

Seos normaal- ja hii-ruut jaotusega

Tõestus

Leiame esmalt murru nimetaja jaotuse (tihedusfunktsiooni).

$$W = \sqrt{Y/f}$$

$$F_W(w) = P(W \le w) = P\left(\sqrt{Y/f} \le w\right)$$
$$= P\left(Y/f \le w^2\right) = P\left(Y \le f\,w^2\right) = F_Y\left(f\,w^2\right)$$

$$f_W(w) = \frac{\partial F_W(w)}{\partial w} = \frac{\partial F_Y\left(f\,w^2\right)}{\partial w} = f_Y\left(f\,w^2\right)\,2fw$$

Seos normaal- ja hii-ruut jaotusega

Tõestus

$$f(x) = \frac{1}{2^{f/2}\Gamma\left(\frac{f}{2}\right)} x^{\frac{f}{2}-1} e^{-\frac{x}{2}},\ x \ge 0$$

Leiame esmalt murru

$$W = \sqrt{Y/f}$$

$$F_W(w) = P(W \le w) = P\left(\sqrt{Y/f} \le w\right)$$
$$= P\left(Y/f \le w^2\right) = P\left(Y \le f\,w^2\right) = F_Y\left(f\,w^2\right)$$

$$f_W(w) = \frac{\partial F_W(w)}{\partial w} = \frac{\partial F_Y\left(f\,w^2\right)}{\partial w} = f_Y\left(f\,w^2\right)\,2fw$$
$$= \frac{1}{2^{f/2}\Gamma\left(\frac{f}{2}\right)}(fw^2)^{\frac{f}{2}-1} e^{-\frac{fw^2}{2}} \cdot 2fw$$

Seos normaal- ja hii-ruut jaotusega

$$Z = \frac{X}{\sqrt{Y/f}} = \frac{X}{W} \qquad f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$f_W(w) = \frac{1}{2^{f/2}\Gamma\left(\frac{f}{2}\right)}(fw^2)^{\frac{f}{2}-1} e^{-\frac{fw^2}{2}} \cdot 2fw$$

$$f_Z(z) = \int_{-\infty}^{\infty} |t| f_W(t) f_X(zt)\,dt$$
$$= \int_{-\infty}^{\infty} |t| f_W(t) \frac{1}{\sqrt{2\pi}} e^{-(zt)^2/2}\,dt$$
$$= \int_{0}^{\infty} t \cdot \frac{1}{2^{f/2}\Gamma\left(\frac{f}{2}\right)}(ft^2)^{\frac{f}{2}-1} e^{-\frac{ft^2}{2}} 2ft \cdot \frac{1}{\sqrt{2\pi}} e^{-(zt)^2/2}\,dt$$

Seos normaal- ja hii-ruut jaotusega

$$f_Z(z) = \int_{0}^{\infty} t \cdot \frac{1}{2^{f/2}\Gamma\left(\frac{f}{2}\right)}(ft^2)^{\frac{f}{2}-1} e^{-\frac{ft^2}{2}} 2ft \cdot \frac{1}{\sqrt{2\pi}} e^{-(zt)^2/2}\,dt$$
$$= \frac{1}{\sqrt{\pi}\Gamma\left(\frac{f}{2}\right)} \int_{0}^{\infty} \frac{ft^2}{2^{(f-1)/2}} (ft^2)^{\frac{f}{2}-1} e^{-\frac{t^2}{2}(f+z^2)}\,dt$$

$$\Gamma(z) = \int_0^\infty y^{z-1} e^{-y} dy.$$

$$= \frac{1}{\sqrt{\pi}\Gamma\left(\frac{f}{2}\right)} \int_{0}^{\infty} (ft^2)^{1/2} (ft^2/2)^{\frac{f+1}{2}-1} e^{-\frac{t^2}{2}(f+z^2)}\,dt$$
$$= \frac{\sqrt{f}}{\sqrt{\pi}\Gamma\left(\frac{f}{2}\right)} \int_{0}^{\infty} \left(\frac{f}{f+z^2}\right)^{\frac{f+1}{2}-1} \left(\frac{(f+z^2)t^2}{2}\right)^{\frac{f+1}{2}-1} e^{-\frac{t^2}{2}(f+z^2)} t\,dt$$

Seos normaal- ja hii-ruut jaotusega

$$\Gamma(z) = \int_0^{\infty} y^{z-1} e^{-y} dy.$$

$$f_Z(z) = \frac{\sqrt{f}}{\sqrt{\pi}\Gamma\left(\frac{f}{2}\right)} \int_0^{\infty} \left(\frac{f}{f+z^2}\right)^{\frac{f+1}{2}-1} \left(\frac{(f+z^2)t^2}{2}\right)^{\frac{f+1}{2}-1} e^{-\frac{t^2}{2}(f+z^2)} t\,dt$$

$$d\left(\frac{t^2}{2}(f+z^2)\right) = (f+z^2)t\,dt \qquad t\,dt = \frac{1}{f}\frac{f}{f+z^2}\,d\left(\frac{t^2}{2}(f+z^2)\right)$$

$$= \frac{1}{\sqrt{f\pi}\Gamma\left(\frac{f}{2}\right)} \int_0^{\infty} \left(\frac{f}{f+z^2}\right)^{\frac{f+1}{2}} \left(\frac{(f+z^2)t^2}{2}\right)^{\frac{f+1}{2}-1} e^{-\frac{t^2}{2}(f+z^2)}\, d\left(\frac{t^2}{2}(f+z^2)\right)$$

$$= \frac{1}{\sqrt{f\pi}\Gamma\left(\frac{f}{2}\right)} \left(\frac{f}{f+z^2}\right)^{\frac{f+1}{2}} \Gamma\left(\frac{f+1}{2}\right) \qquad = \frac{\Gamma\left(\frac{f+1}{2}\right)}{\sqrt{f\pi}\Gamma\left(\frac{f}{2}\right)} \left(\frac{f+z^2}{f}\right)^{-\frac{f+1}{2}}$$

Seos normaaljaotusega

$$\lim_{f\to\infty} \frac{\Gamma\left(\frac{f+1}{2}\right)}{\sqrt{f\pi}\ \Gamma\left(\frac{f}{2}\right)} \cdot \left(1+\frac{x^2}{f}\right)^{-\frac{f+1}{2}} = ?$$

$$\lim_{f\to\infty} \left(1+\frac{x^2}{f}\right)^{-\frac{f+1}{2}} = \lim_{f\to\infty} \exp\left\{\ln\left[\left(1+\frac{x^2}{f}\right)^{-\frac{f+1}{2}}\right]\right\}$$

$$\lim_{x\to a} a^{f(x)} = a^{\lim_{x\to a} f(x)}$$

$$= \exp\left\{\lim_{f\to\infty} \ln\left[\left(1+\frac{x^2}{f}\right)^{-\frac{f+1}{2}}\right]\right\} = \exp \lim_{f\to\infty} -\frac{\ln\left(1+\frac{x^2}{f}\right)}{2/(f+1)}$$

Seos normaaljaotusega

$$\lim_{f\to\infty} -\frac{\ln\left(1+\frac{x^2}{f}\right)}{2/(f+1)} = \lim_{f\to\infty} -\frac{f/(f+x^2)\cdot(-1)\cdot x^2/f^2}{(-1)\cdot 2/(f+1)^2}$$

$$= \lim_{f\to\infty} -\frac{(f+1)^2}{f(f+x^2)}\cdot\frac{x^2}{2} = -\frac{x^2}{2}$$

$$\lim_{f\to\infty} \left(1+\frac{x^2}{f}\right)^{-\frac{f+1}{2}} = \exp\left\{\lim_{f\to\infty} \ln\left[\left(1+\frac{x^2}{f}\right)^{-\frac{f+1}{2}}\right]\right\}$$

$$= \exp\left(-\frac{x^2}{2}\right)$$

$$\lim_{f\to\infty} \frac{\Gamma\left(\frac{f+1}{2}\right)}{\sqrt{f\pi}\ \Gamma\left(\frac{f}{2}\right)} \cdot \left(1+\frac{x^2}{f}\right)^{-\frac{f+1}{2}}$$

$$= \exp\left(-\frac{x^2}{2}\right) \lim_{f\to\infty} \frac{\Gamma\left(\frac{f+1}{2}\right)}{\sqrt{f\pi}\ \Gamma\left(\frac{f}{2}\right)}$$

Wikipediast: $\lim_{n\to\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n)n^{\alpha}} = 1$

$$\lim_{f\to\infty} \frac{\Gamma(f/2+1/2)}{\Gamma(f/2)(f/2)^{1/2}} = 1$$

$$\lim_{f\to\infty} \frac{\Gamma(f/2+1/2)\sqrt{2}}{\Gamma(f/2)\sqrt{f}} = 1$$

$$= \exp\left(-\frac{x^2}{2}\right) \frac{1}{\sqrt{2\pi}} \lim_{f\to\infty} \frac{\Gamma\left(\frac{f+1}{2}\right)\sqrt{2}}{\sqrt{f}\ \Gamma\left(\frac{f}{2}\right)} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

Oluline erijuht: *t*(1)

$$f(x) = \frac{\Gamma\left(\frac{f+1}{2}\right)}{\sqrt{f\pi}\ \Gamma\left(\frac{f}{2}\right)} \cdot \left(1 + \frac{x^2}{f}\right)^{-\frac{f+1}{2}}.$$

$$f(x) = \frac{1}{\sqrt{\pi}\sqrt{\pi}} \cdot \left(1 + x^2\right)^{-1} = \frac{1}{\pi(1+x^2)}$$

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)}\, dx$$

Definitsioon 12.3

Defineeritakse

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_{c}^{\infty} f(x)\, dx, \quad \text{kus } c \in \mathbb{R} \text{ on suvaline.}$$

,

Kui vähemalt üks integraalidest piirkondades $(-\infty, c]$ ja $[c, \infty)$ hajub, siis integraal piirkonnas $(-\infty, \infty)$ hajub.

Oluline erijuht: *t*(1)

$$f(x) = \frac{\Gamma\left(\frac{f+1}{2}\right)}{\sqrt{f\pi}\ \Gamma\left(\frac{f}{2}\right)} \cdot \left(1 + \frac{x^2}{f}\right)^{-\frac{f+1}{2}}.$$

$$f(x) = \frac{1}{\sqrt{\pi}\sqrt{\pi}} \cdot \left(1 + x^2\right)^{-1} = \frac{1}{\pi(1+x^2)}$$

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)}\, dx$$

$$= \int_{-\infty}^{0} \frac{x}{\pi(1+x^2)}\, dx + \int_{0}^{\infty} \frac{x}{\pi(1+x^2)}\, dx$$

Oluline erijuht: *t*(1)

$$f(x) = \frac{1}{\pi(1+x^2)}$$

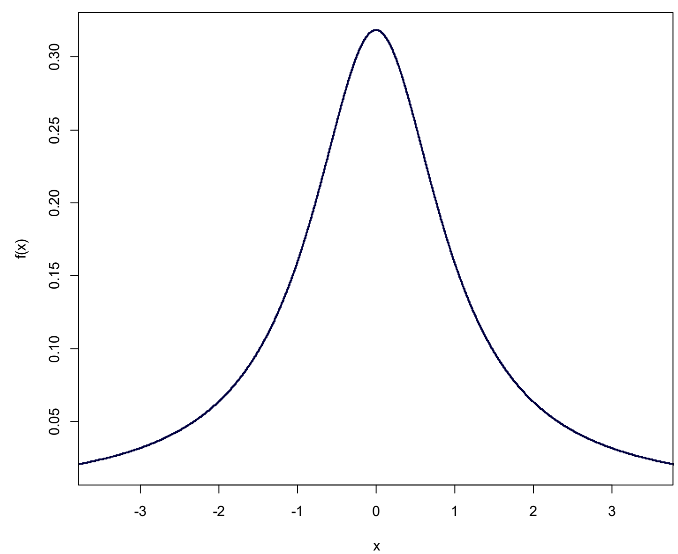
$$\mathrm{E}(X) = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)}\, dx$$

$$= \underbrace{\int_{-\infty}^{0} \frac{x}{\pi(1+x^2)}\, dx}_{-\infty} + \underbrace{\int_{0}^{\infty} \frac{x}{\pi(1+x^2)}\, dx}_{\infty}$$

$$\int_{0}^{\infty} \frac{x}{\pi(1+x^2)}\, dx = \int_{1}^{\infty} \frac{1}{2\pi(1+x^2)}\, d(1+x^2)$$

$$= \frac{1}{2\pi} \ln(x)\Big|_{1}^{\infty} = \infty$$

t(1) või (standardne) Cauchy jaotus

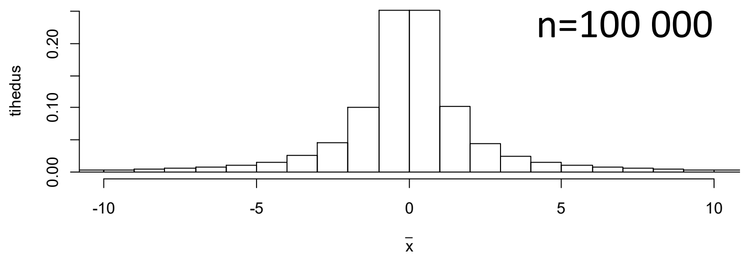


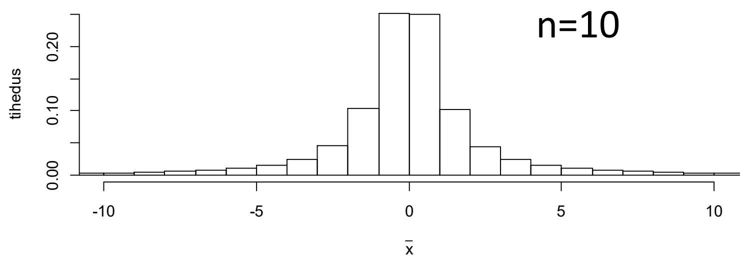
mediaan=0

sümmeetriline: $f(x)=f(-x)$

Keskväärtus ei ole 0!

valimikeskmise jaotus





t-jaotus üldjuhul

sümmeetriline 0-punkti suhtes

Mediaan: 0

Keskväärtus: 0, kui f>1
defineerimata, kui f=1

Dispersioon: $f/(f$-2), kui f>2
∞, kui f=2
defineerimata, kui f=1

Teoreem (Keskväärtuse ja standardhälbe jagatisest)

Olgu $X_i \sim N(\mu, \sigma)$ sõltumatud juhuslikud suurused, $i = 1, 2, \ldots, n$, siis

$$\frac{\bar{X} - \mu}{\boldsymbol{s}/\sqrt{n}} \sim t(n-1),$$

kus $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ ja $\boldsymbol{s} = (\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2)^{1/2}$.

Tõestus

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0;1) \qquad \frac{n-1}{\sigma^2}s^2 \sim \chi^2(n-1)$$

$$X \sim N(0;1);\ Z \sim \chi^2(f);\ X \perp Z$$

$$Z = \frac{X}{\sqrt{Y/f}} \sim t(f).$$

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \cdot \frac{\sqrt{\sigma^2}}{\sqrt{(n-1)s^2/(n-1)}} = \frac{\overline{X} - \mu}{s/\sqrt{n}} \sim t(n-1)$$

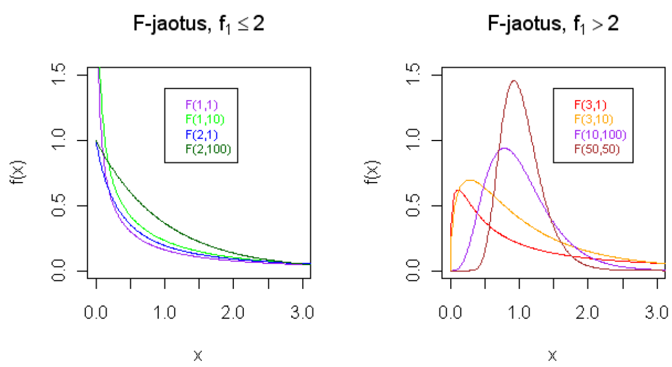
Definitsioon

Juhuslik suurus X on $F-$jaotusega vabadusastmete arvudega f_1 ja f_2, $X \sim F(f_1, f_2)$, kui tema tihedusfunktsioon avaldub seosega

$$f(x) = k\ x^{1/2(f_1-2)}(f_2 + f_1 x)^{-1/2(f_1+f_2)}, \quad x > 0,$$

kus normeeriv konstant omab kuju

$$k = \frac{f_1^{f_1/2} f_2^{f_2/2}\Gamma(\frac{f_1+f_2}{2})}{\Gamma(\frac{f_1}{2})\Gamma(\frac{f_2}{2})}.$$



- tihedusfunktsiooni kuju on langev, kui $f_1 \leq 2$;
- kui $f_1 > 2$, siis on tegemist ühemodaalse ebasümmeetrilise jaotusega;
- keskväärtus eksisteerib, kui $f_2 > 2$: $EX = \frac{f_2}{f_2 - 2}$;
- dispersioon eksisteerib, kui $f_2 > 4$: $DX = \frac{2f_2^2(f_1+f_2-2)}{f_1(f_2-2)^2(f_2-4)}$.

Teoreem (Seos hii-ruut jaotusega)

Kui juhuslik suurus $U \sim \chi^2(f_1)$ ja $V \sim \chi^2(f_2)$ ning U ja V on sõltumatud, siis juhuslik suurus X on F- jaotusega:

$$X = \frac{U/f_1}{V/f_2} \sim F(f_1, f_2).$$

Kui X~F(f_1; f_2), siis 1/X~F(f_2; f_1)

Teoreem (Kahe valimidispersiooni jagatisest)

Olgu antud juhuslik valim $x_1, x_2, ..., x_{n_1}$ jaotusest $N(\mu_1, \sigma_1)$ ja sellest sõltumatu valim $y_1, y_2, ..., y_{n_2}$ jaotusest $N(\mu_2, \sigma_2)$. Vastavad valimite dispersioonid olgu s_1 ja s_2. Siis

$$\frac{\mathbf{s}_1^2/\sigma_1^2}{\mathbf{s}_2^2/\sigma_2^2} \sim F(n_1 - 1, n_2 - 1),$$

kus $\mathbf{s}_i^2$, $i = 1, 2$ on valimidispersioonile $s_i^2, i = 1, 2$ vastav hinnangufunktsioon.