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The normal inverse Gaussian distribution: Exposition and applications

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Talk outline

Why fit closing prices with distributions?

Specific NIG qualities.

Selecting best fit models

Analysis of data.

Applications.

The purpose of the study

- Purpose of the study is to model stock price distributions by Normal Inverse Gaussian distribution (NIG).
- NIG distributions form a 4-parameter density family.
- We consider **stock prices** (companies trading on Tallinn Stock Exchange: 01/01/2008 – 01/01/2012), **world indexes** (US data: 21/04/2004 – 29/12/2011) and **exchange rates** (UK data: 12/04/2008 – 07/08/2012)

Construction of NIG (1)

Consider a bivariate Brownian motion (u_t, v_t) starting at point $(u, 0)$ and having constant drift vector (β, γ) with $\gamma > 0$ and let *z* denote the time at which v_t hits the line $v = \delta > 0$ for the first time (u_t, v_t are assumed independent).

Then letting $\alpha = \sqrt{\beta^2 + \gamma^2}$, the law of u_z is $NIG(\alpha, \beta, \delta, \mu)$

Consider a bivariate Brownian motion (u_t, v_t) starting at point $(u, 0)$ and having constant drift vector (β, γ) with $\gamma > 0$ and let *z* denote the time at which v_t hits the line $v = \delta > 0$ for the first time (u_t, v_t are assumed independent).
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Construction of NIG (2)

$NIG(\alpha, \beta, \delta, \mu)$ distribution can be defined as a normal variance-mean mixture i.e. it can be presented as the marginal distribution of X in the pair (X, Z), where the conditional probability X|Z is given by

$$X|Z = z \sim N(\mu + \beta z, z)$$

where ($\sim$ means is distributed as) the variable $Z\ (z > 0) \sim\ F_z$ and

$$F_z = \Phi\left(\frac{1}{\sqrt{z}}\left[z\sqrt{\alpha^2 - \beta^2} - \delta\right]\right) + \ exp\left\{2\delta\sqrt{\alpha^2 - \beta^2}\right\}\ \Phi\left(\frac{-1}{\sqrt{z}}\left[z\sqrt{\alpha^2 - \beta^2} + \delta\right]\right)$$

with $\Phi(z) \sim N(0,1)$.

$NIG(\alpha, \beta, \delta, \mu)$ distribution can also be constructed through the general hyperbolic class of distribution; in the special case where $\lambda = {}^{-1}/_{2}$. This distribution can be reduced to the Chi-square distribution.

NIG Distribution

PDF: $f_{NIG}(x\colon \alpha,\beta,\delta,\mu) = \frac{\alpha\delta}{\pi} e^{\left(\delta(\alpha^2-\beta^2)^{1/2}-\beta(x-\mu)\right)} \frac{K_1\left(\alpha(\delta^2+(x-\mu)^2)^{1/2}\right)}{(\delta^2-(x-\mu)^2)^{1/2}}$

where $K_1(x) = \frac{x}{4}\int_0^{\infty} e^{\left(t+\frac{y^2}{4t}\right)} t^{-2}\, dt$, $\boldsymbol{\delta > 0}$ **is scale parameter**, $\boldsymbol{\alpha > 0}$ **is tail heaviness**, $\boldsymbol{\beta \geq 0}$ **is symmetry parameter** with condition $-\alpha < \beta < \alpha$ and $\boldsymbol{\mu}$ **is location parameter**.

MGF: $M(x\colon \alpha,\beta,\delta,\mu) = exp\left[\delta\left\{\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+x)^2}\right\} + x\mu\right]$, x > 0

- **Mean** = $\mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}$, **Variance** = $\alpha^2\delta(\alpha^2-\beta^2)^{-3/2}$
- **Skewness** = $3\beta\alpha^{-1}\delta^{-1}(\alpha^2-\beta^2)^{-1/4}$, **Kurtosis** = $3\left(1+\frac{\alpha^2+4\beta^2}{\delta\alpha^2\sqrt{\alpha^2-\beta^2}}\right)$

More NIG useful characteristics

- If X ~ $NIG(\alpha, \beta, \delta, \mu)$, the $\mathrm{Y} = \mathrm{kX} \sim NIG(\alpha/k, \beta/k, \delta/k, \mu/k)$ - scaling
- $K_1(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$, **semi heavy-tail** able to capture tails of distributions

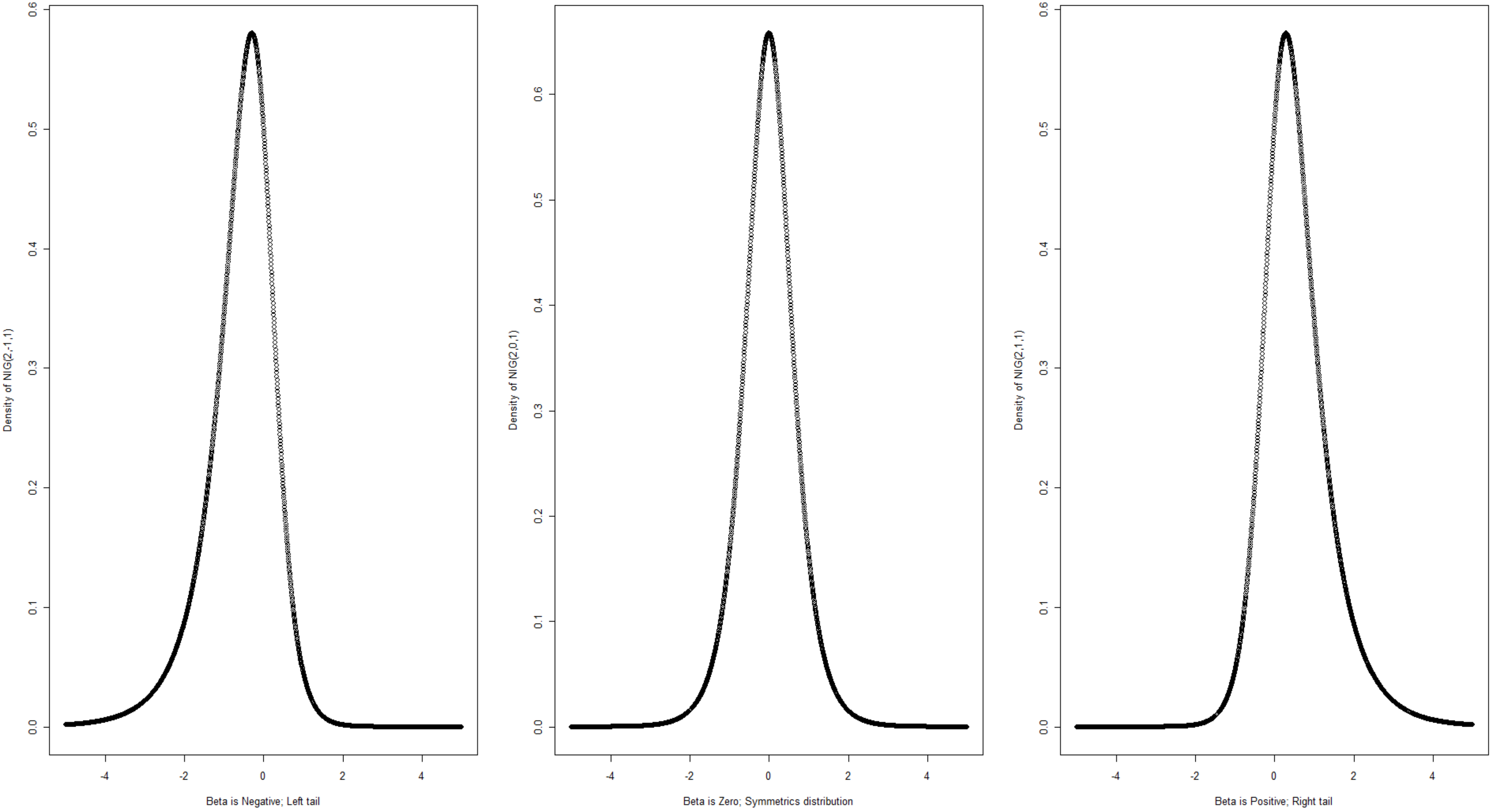
- If $X_1 \sim NIG(\alpha, \beta, \delta_1, \mu_1)$ and $X_2 \sim NIG(\alpha, \beta, \delta_2, \mu_2)$ are

independent, then Y = $X_1 + X_2 \sim NIG(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2)$

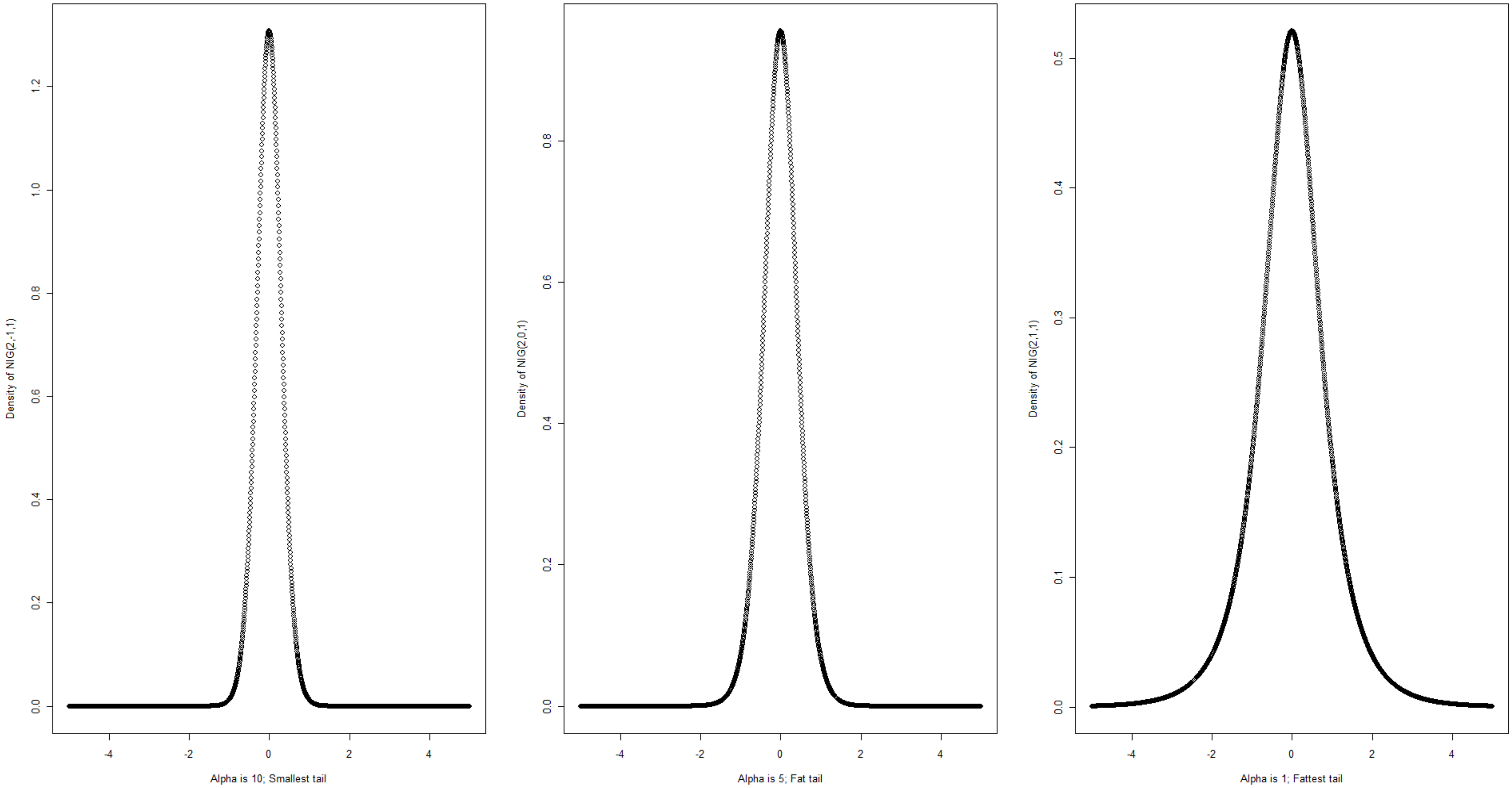
or

if $x_1, x_2, x_3, \dots, x_m$ are independent normal inverse Gaussian random numbers with common parameters α, β but having individual location and scale parameters u_i and $\delta_i (i = 1, \dots, m)$, then $x_+ = x_1 + \cdots + x_m$ is again distributed according to a normal inverse Gaussian law, with parameters $(\alpha, \beta, \delta_+, u_+)$

NIG density for negative, zero and positive beta: Other parameters held constant



NIG density for alpha=10, 5 and 1: Other parameters held constant



Method for selecting best models

1. choose a suitable class of distributions (using general or prior information about the specific data) ;

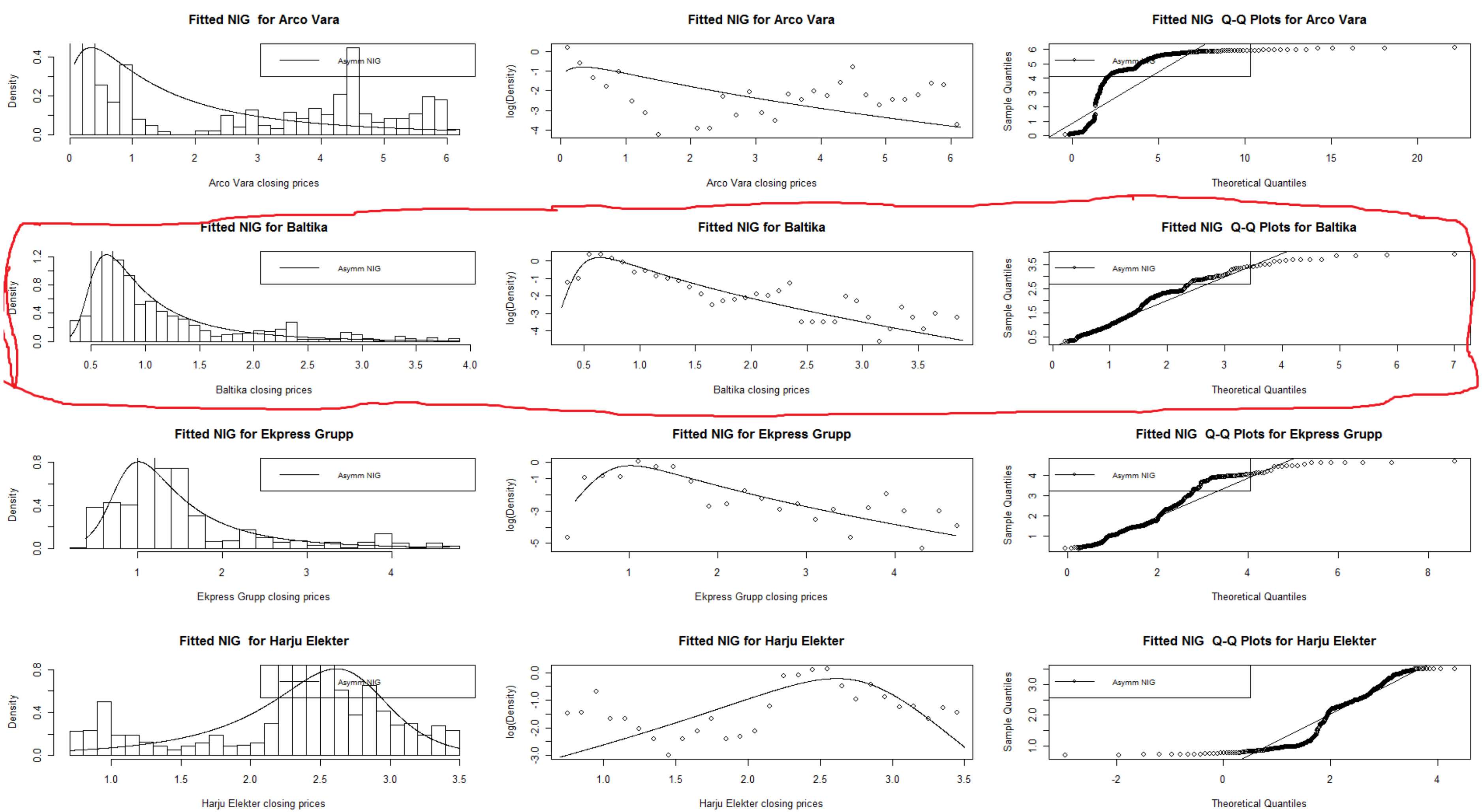
2. estimate the parameters (by finding maximum likelihoods);

3. estimate goodness of fit;

- a) visual estimation,
- b) classical goodness-of-fit tests (Kolmogorov-Smirnov, Chi-squared with equiprobable classes),
- c) probability or quantile-quantile plots.

1) Visual estimation: Estonian companies

(Data from Tallinn Stock Exchange: 01/01/2008 – 01/01/2012)



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Estimated NIG parameters, skews, kurtoses, Kolmogorov-Smirnov (K-S) and Chi-square (χ2) test results for Estonian companies

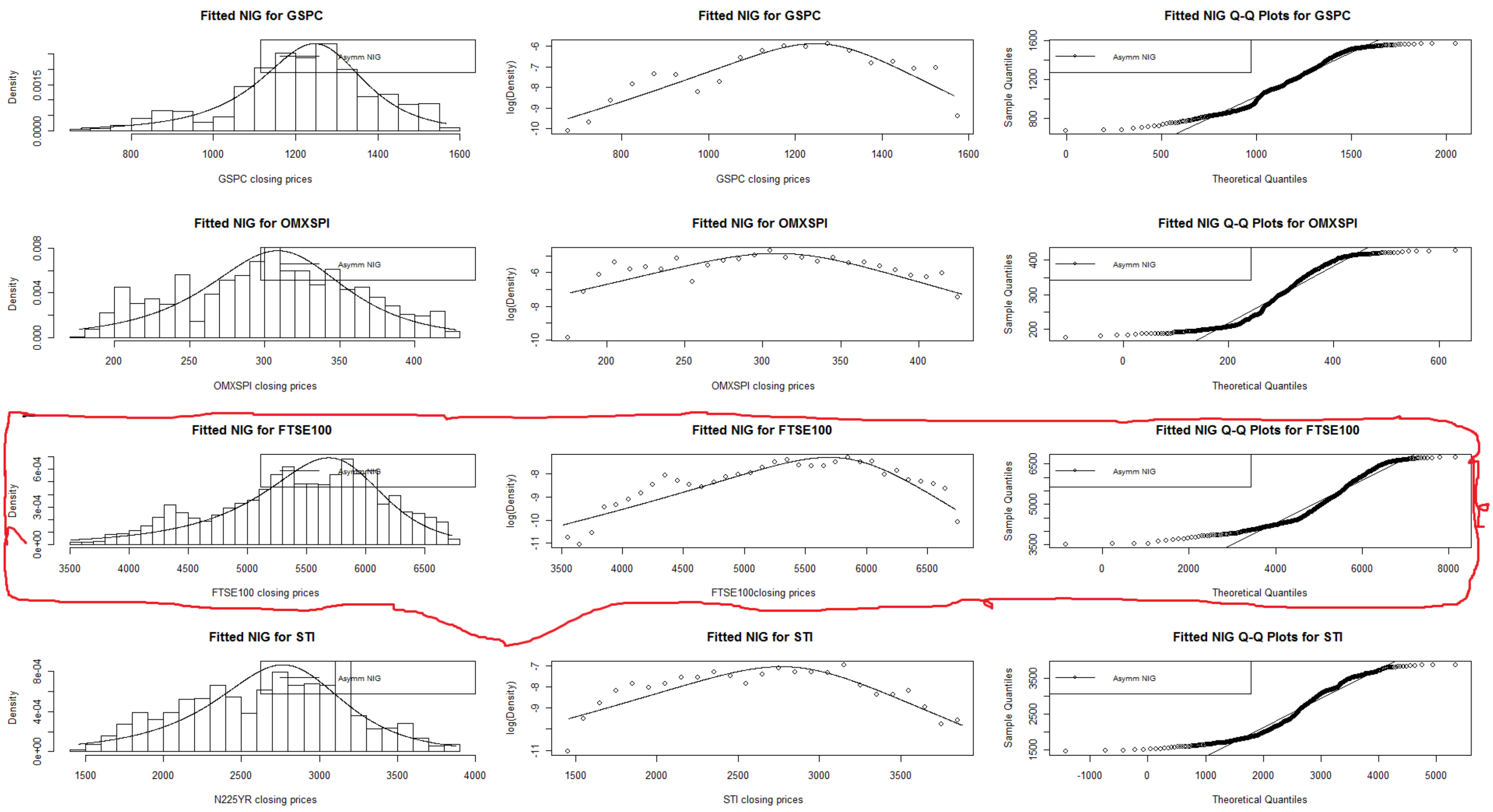
(Data from Tallinn Stock Exchange: 01/01/2008 – 01/01/2012)

Company	Alpha	Beta	Delta	mu
Arco Vara	468,90	468,86	0,03	0,02
Baltika	7,06	6,62	0,22	0,52
Ekpress Grupp	2,68	2,15	0,49	0,85
Harju Elekter	3,20	-2,07	0,72	2,95

Company	χ2 statistic	χ2 p-value	K-S D-value	K-S p-value	Skew	Kurtosis
Arco Vara	2251,60	P < 0,00001	0,23	p < 0,00001	0.38	-1.53
Baltika	1771,12	P < 0,00001	0,06	p = 0,05723	1.67	2.33
Ekpress Grupp	1194,24	P < 0,00001	0,07	P = 0,01198	1.70	2.53
Harju Elekter	1345,87	P < 0,00001	0,09	p = 0,00027	-0.82	-0.05

2) Visual estimation: World indexes

(Data obtained from US: 21/04/2004 – 29/12/2011)



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Estimated NIG parameters, skews, kurtoses, Kolmogorov-Smirnov (K-S) and Chi-square (χ2) test results for world indexes (Data obtained from US: 21/04/2004 – 29/12/2011)

World Index	Alpha	Beta	Delta	Mu	Skew	Kurtosis	K-S p-value	K-S D-value	χ2 Statistic	χ2 p-value
GSPC	0.04	-0.03	514.04	1667.66	-0.05	0.12	P=0.016	0.015	36.49	1
OMXSPI	0.93	-0.70	870.55	1311.78	-0.06	-0.76	P=0.002	0.06	122.57	1
FTSE100	0.03	-0.028	1195.17	7947.69	-3.92	-5.5	P=0.097	0.04	82.62	1
STI	0.93	-0.7	870.55	1311.78	-0.06	-0.76	P=0.001	0.053	122.57	1

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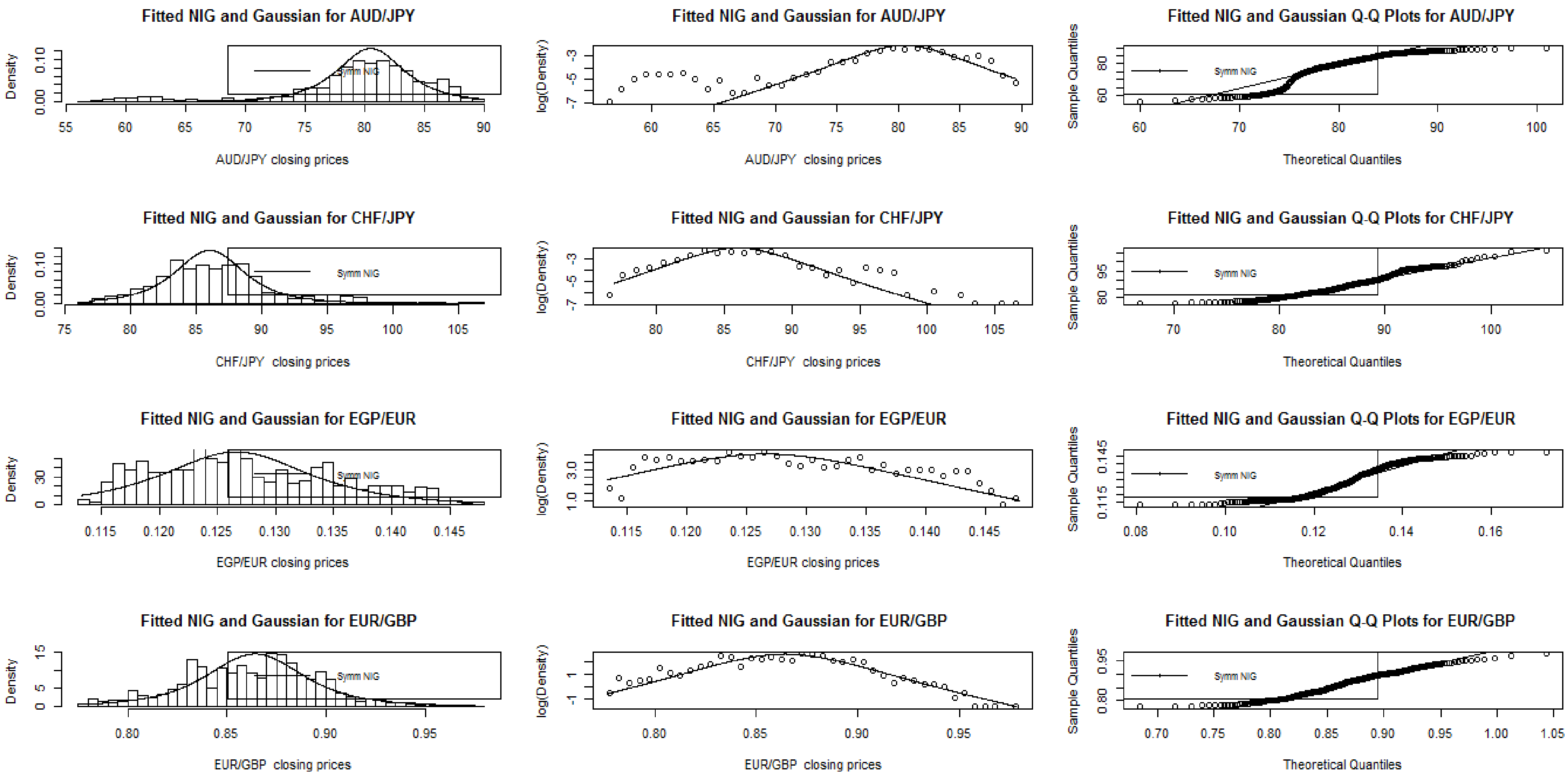
3) Visual estimation: Exchange rates

(Data obtained from UK: 12/04/2008 – 07/08/2012)



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3) Visual estimation: Exchange rates

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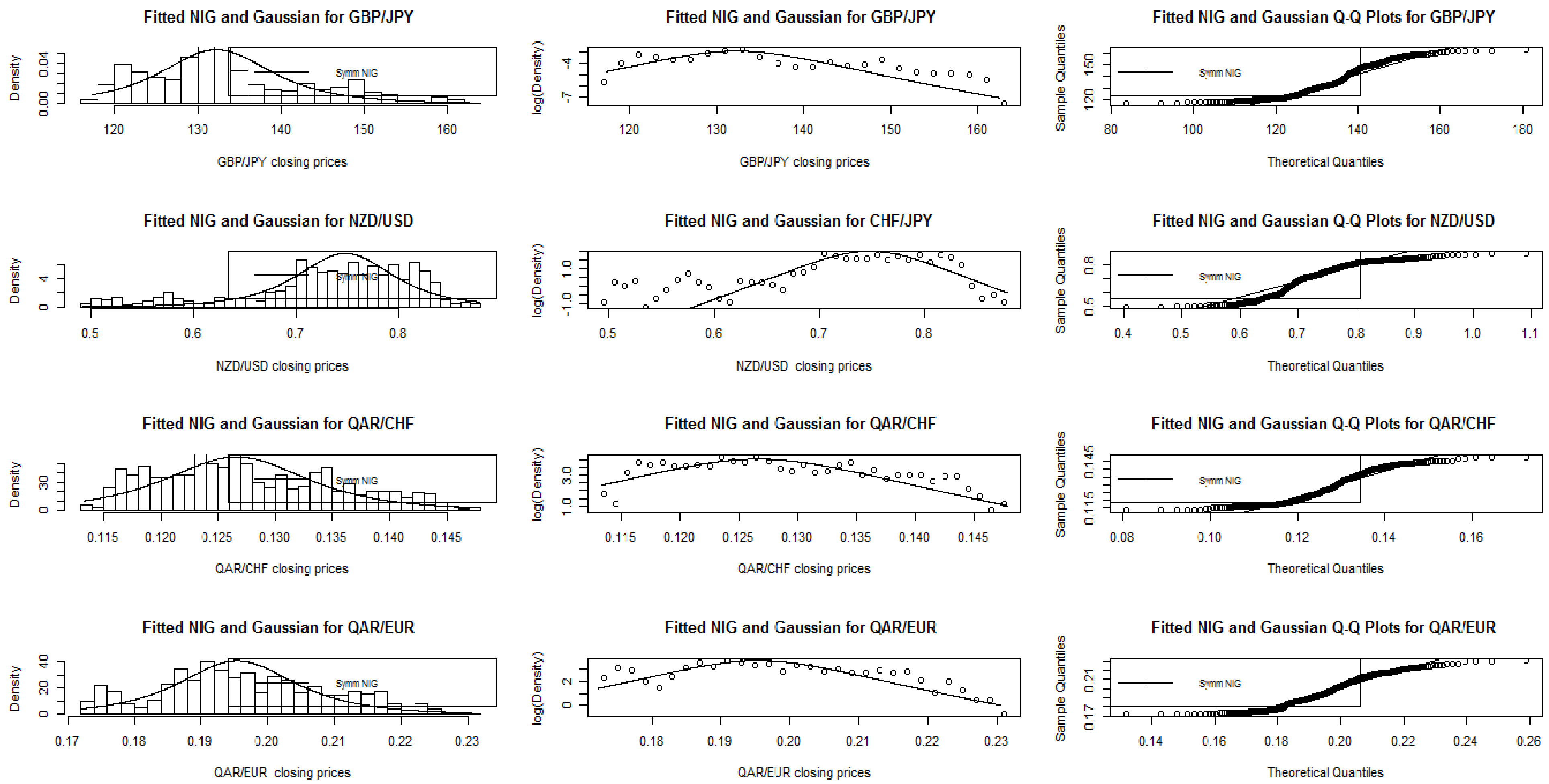

3) Visual estimation: Exchange rates

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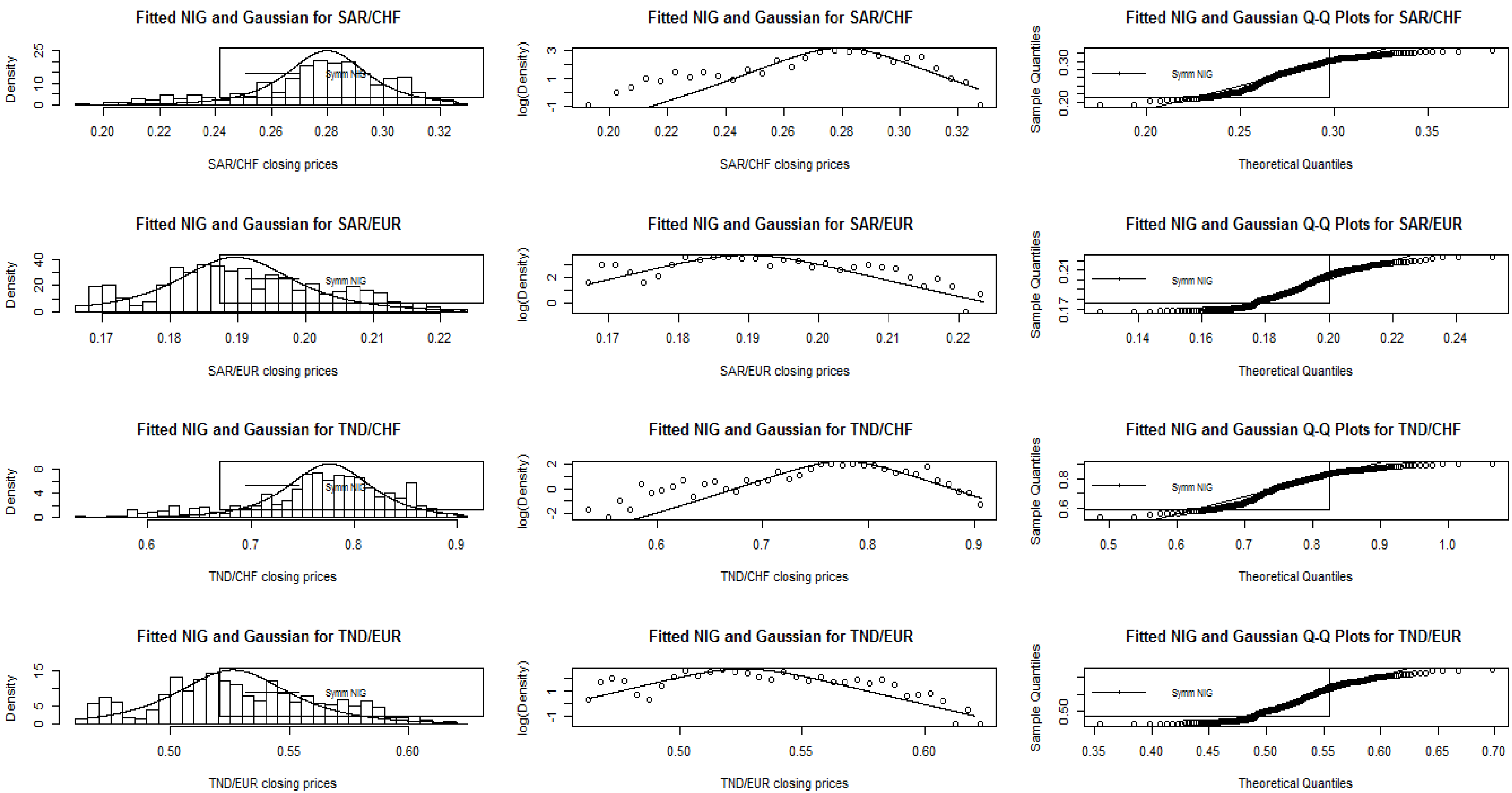

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Estimated NIG parameters, skews, kurtoses, Kolmogorov-Smirnov (K-S) and Chi-square (χ2) test results for NIG FX models (Data obtained from UK: 12/04/2008 – 07/08/2012)

FX	Alpha	Beta	Delta	Mu	Skew	Kurtosis	KS p-value	KS D-value	CS P-value	CS stats
AUD/JPY	0.33	-0.23	5.07	84.3	-1.53	2.49	0.4	0.04	1	158.1
CHF/JPY	0.54	0.26	7.31	82.41	0.76	1.16	0.22	0.047	1	169.28
EGP/EUR	18215.6	18011.2	0.004	0.1	0.38	-0.76	0.062	0.059	0.01	350.14
EUR/GBP	2194.25	-412.12	2.41	1.32	-0.01	-0.07	0.08	0.057	1	227.74
GBP/JPY	8.43	8.31	4.32	108.3	0.65	-0.3	0.12	0.054	1	100.5
NZD/USD	354.97	-342.22	0.048	0.91	-0.98	0.44	0.24	0.046	1	103.31
QAR/CHF	2152.2	-2092.4	0.02	0.37	-0.77	0.29	0.12	0.053	1	304.05
QAR/EUR	1364.5	1022.2	0.07	0.12	0.19	-0.6	0.37	0.041	0.963	359.56
SAR/CHF	2656.5	-2594.4	0.02	0.36	-0.77	0.28	0.16	0.05	1	329.36
SAR/EUR	2702.86	2331.53	0.054	0.099	0.19	-0.6	0.16	0.05	0.92	359
TND/CHF	1088.3	-1065.6	0.047	0.99	-0.76	0.28	0.341	0.042	1	169.02
TND/EUR	1014.79	878.64	0.153	0.27	0.18	-0.6	0.341	0.042	1	254.9

Observations…

Daily closing prices (12/04/2008 – 07/08/2012) of AUD/JPY, CHF/JPY, GBP/JPY, NZD/USD, QAR/CHF, SAR/CHF, SAR/EUR, TND/CHF, TND/EUR are excellent fits while EGP/EUR and EUR/GBP are good fits with a Kolmogorov-Smirnov test p-value of 0.062 and 0.08 respectively.

Impossible to estimate NIG parameters (by maximum likelihood) for JPY/CHF but CHF/JPY was an excellent fit.

Thus, while the stochastic properties of an exchange rate can be completely modeled with a probability distribution in one direction, it may be impossible the other way around.

Impossible to estimate	Bad models				
AUD/USD	USD/EUR	CHF/EUR	CHF/GBP	TND/JPY	TND/GBP
CHF/USD	EGP/USD	EUR/JPY	TND/GBP	JOD/GBP	SAR/USD
EUR/JOD	JOD/JPY	JPY/EUR	JPY/GBP	EGP/CHF	USD/JOD
JPY/CHF	CAD/JPY	QAR/GBP	QAR/USD	USD/JPY	XAU/USD
JPY/USD			TND/JPY	TND/USD	USD/GBP

Major conclusions

- The distribution of closing prices of stocks, world indexes and exchange rates can be modeled with the normal inverse Gaussian distribution; despite different time horizons
- Modeling the distribution of an exchange rate in one direction does not mean it can be modeled in the other direction

N/B: Chi-square test results depend on how intervals are chosen, number of variables etc and can basically be ignored in our study.

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Thank you for listening

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