The normal inverse Gaussian distribution: Exposition and applications

Dean Teneng, Institute of Mathematical Statistics, University of Tartu, Estonia.

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Talk outline

Why fit closing prices with distributions?
Specific NIG qualities.
Selecting best fit models
Analysis of data.
Applications.
The purpose of the study

• Purpose of the study is to model stock price distributions by Normal Inverse Gaussian distribution (NIG).

• NIG distributions form a 4-parameter density family.

• We consider stock prices (companies trading on Tallinn Stock Exchange: 01/01/2008 – 01/01/2012), world indexes (US data: 21/04/2004 – 29/12/2011) and exchange rates (UK data: 12/04/2008 – 07/08/2012)
Construction of NIG (1)

Consider a bivariate Brownian motion \((u_t, v_t)\) starting at point \((u, 0)\) and having constant drift vector \((\beta, \gamma)\) with \(\gamma > 0\) and let \(z\) denote the time at which \(v_t\) hits the line \(v = \delta > 0\) for the first time \((u_t, v_t)\) are assumed independent.

Then letting \(\alpha = \sqrt{\beta^2 + \gamma^2}\), the law of \(u_z\) is NIG\((\alpha, \beta, \delta, \mu)\)
Construction of NIG (2)

\( NIG(\alpha, \beta, \delta, \mu) \) distribution can be defined as a normal variance-mean mixture i.e. it can be presented as the marginal distribution of \( X \) in the pair \((X, Z)\), where the conditional probability \( X|Z \) is given by

\[
X|Z = z \sim N(\mu + \beta z, z)
\]

where (\( \sim \) means is distributed as) the variable \( Z (z > 0) \sim F_z \) and

\[
F_z = \Phi \left( \frac{1}{\sqrt{z}} \left[ z \sqrt{\alpha^2 - \beta^2} - \delta \right] \right) + \exp \left\{ 2\delta \sqrt{\alpha^2 - \beta^2} \right\} \Phi \left( \frac{-1}{\sqrt{z}} \left[ z \sqrt{\alpha^2 - \beta^2} + \delta \right] \right)
\]

with \( \Phi(z) \sim N(0, 1) \).

\( NIG(\alpha, \beta, \delta, \mu) \) distribution can also be constructed through the general hyperbolic class of distribution; in the special case where \( \lambda = -1/2 \). This distribution can be reduced to the Chi-square distribution.
NIG Distribution

PDF: \( f_{NIG}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha \delta}{\pi} e^{\left(\delta (\alpha^2 - \beta^2)^{1/2} - \beta (x - \mu)\right)} \frac{K_1\left(\frac{\alpha (\delta^2 + (x - \mu)^2)^{1/2}}{(\delta^2 - (x - \mu)^2)^{1/2}}\right)}{(\delta^2 - (x - \mu)^2)^{1/2}} \)

where \( K_1(x) = \frac{\chi}{4} \int_0^\infty e^{(t + \frac{y^2}{4t})} t^{-2} dt \), \( \delta > 0 \) is scale parameter, \( \alpha > 0 \) is tail heaviness, \( \beta \geq 0 \) is symmetry parameter with condition \(-\alpha < \beta < \alpha\) and \( \mu \) is location parameter.

MGF: \( M(x; \alpha, \beta, \delta, \mu) = \exp\left[\delta \left\{\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + x)^2}\right\} + x\mu\right], \ x > 0 \)

- Mean = \( \mu + \frac{\delta \beta}{\sqrt{\alpha^2 - \beta^2}} \), \ Variance = \( \alpha^2 \delta (\alpha^2 - \beta^2)^{-3/2} \)
- Skewness = \( 3\beta \alpha^{-1} \delta^{-1} (\alpha^2 - \beta^2)^{-1/4} \), \ Kurtosis = \( 3\left(1 + \frac{\alpha^2 + 4\beta^2}{\delta \alpha^2 \sqrt{\alpha^2 - \beta^2}}\right) \)
More NIG useful characteristics

- If $X \sim NIG(\alpha, \beta, \delta, \mu)$, the $Y = kX \sim NIG(\alpha/k, \beta/k, \delta/k, \mu/k)$ - scaling

- $K_1(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$, semi heavy-tail able to capture tails of distributions

- If $X_1 \sim NIG(\alpha, \beta, \delta_1, \mu_1)$ and $X_2 \sim NIG(\alpha, \beta, \delta_2, \mu_2)$ are independent, then $Y = X_1 + X_2 \sim NIG(\alpha, \beta, \delta_1 + \delta_2, \mu_1 + \mu_2)$

or

if $x_1, x_2, x_3, \ldots, x_m$ are independent normal inverse Gaussian random numbers with common parameters $\alpha, \beta$ but having individual location and scale parameters $u_i$ and $\delta_i (i = 1, \ldots, m)$, then $x_+ = x_1 + \cdots + x_m$ is again distributed according to a normal inverse Gaussian law, with parameters $(\alpha, \beta, \delta_+, u_+)$
NIG density for negative, zero and positive beta: Other parameters held constant
NIG density for alpha=10, 5 and 1: Other parameters held constant
Method for selecting best models

1. choose a suitable class of distributions (using general or prior information about the specific data) ;
2. estimate the parameters (by finding maximum likelihoods);
3. estimate goodness of fit;
   • a) visual estimation,
   • b) classical goodness-of-fit tests (Kolmogorov-Smirnov, Chi-squared with equiprobable classes),
   • c) probability or quantile-quantile plots.
1) Visual estimation: Estonian companies

(Data from Tallinn Stock Exchange: 01/01/2008 – 01/01/2012)
## Estimated NIG parameters, skews, kurtoses, Kolmogorov-Smirnov (K-S) and Chi-square ($\chi^2$) test results for Estonian companies
(Data from Tallinn Stock Exchange: 01/01/2008 – 01/01/2012)

<table>
<thead>
<tr>
<th>Company</th>
<th>Alpha</th>
<th>Beta</th>
<th>Delta</th>
<th>mu</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arco Vara</td>
<td>468.90</td>
<td>468.86</td>
<td>0.03</td>
<td>0.02</td>
</tr>
<tr>
<td>Baltika</td>
<td>7.06</td>
<td>6.62</td>
<td>0.22</td>
<td>0.52</td>
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<tr>
<td>Ekpress Grupp</td>
<td>2.68</td>
<td>2.15</td>
<td>0.49</td>
<td>0.85</td>
</tr>
<tr>
<td>Harju Elekter</td>
<td>3.20</td>
<td>-2.07</td>
<td>0.72</td>
<td>2.95</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Company</th>
<th>$\chi^2$ statistic</th>
<th>$\chi^2$ p-value</th>
<th>K-S D-value</th>
<th>K-S p-value</th>
<th>Skew</th>
<th>Kurtosis</th>
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</thead>
<tbody>
<tr>
<td>Arco Vara</td>
<td>2251.60</td>
<td>$P &lt; 0.00001$</td>
<td>0.23</td>
<td>$p &lt; 0.00001$</td>
<td>0.38</td>
<td>-1.53</td>
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<tr>
<td>Baltika</td>
<td>1771.12</td>
<td>$P &lt; 0.00001$</td>
<td>0.06</td>
<td>$p = 0.05723$</td>
<td>1.67</td>
<td>2.33</td>
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<tr>
<td>Ekpress Grupp</td>
<td>1194.24</td>
<td>$P &lt; 0.00001$</td>
<td>0.07</td>
<td>$P = 0.01198$</td>
<td>1.70</td>
<td>2.53</td>
</tr>
<tr>
<td>Harju Elekter</td>
<td>1345.87</td>
<td>$P &lt; 0.00001$</td>
<td>0.09</td>
<td>$p = 0.00027$</td>
<td>-0.82</td>
<td>-0.05</td>
</tr>
</tbody>
</table>
2) Visual estimation: World indexes
(Data obtained from US: 21/04/2004 – 29/12/2011)
Estimated NIG parameters, skews, kurtoses, Kolmogorov-Smirnov (K-S) and Chi-square ($\chi^2$) test results for world indexes (Data obtained from US: 21/04/2004 – 29/12/2011)

<table>
<thead>
<tr>
<th>World Index</th>
<th>Alpha</th>
<th>Beta</th>
<th>Delta</th>
<th>Mu</th>
<th>Skew</th>
<th>Kurtosis</th>
<th>K-S p-value</th>
<th>K-S D-value</th>
<th>$\chi^2$ Statistic</th>
<th>$\chi^2$ p-value</th>
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<tbody>
<tr>
<td>GSPC</td>
<td>0.04</td>
<td>-0.03</td>
<td>514.04</td>
<td>1667.66</td>
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<td>P=0.016</td>
<td>0.015</td>
<td>36.49</td>
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<tr>
<td>OMXSPI</td>
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<td>-0.70</td>
<td>870.55</td>
<td>1311.78</td>
<td>-0.06</td>
<td>-0.76</td>
<td>P=0.002</td>
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<td>122.57</td>
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<td>FTSE100</td>
<td>0.03</td>
<td>-0.028</td>
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<td>-5.5</td>
<td>P=0.097</td>
<td>0.04</td>
<td>82.62</td>
<td>1</td>
</tr>
<tr>
<td>STI</td>
<td>0.93</td>
<td>-0.7</td>
<td>870.55</td>
<td>1311.78</td>
<td>-0.06</td>
<td>-0.76</td>
<td>P=0.001</td>
<td>0.053</td>
<td>122.57</td>
<td>1</td>
</tr>
</tbody>
</table>
3) Visual estimation: Exchange rates
(Data obtained from UK: 12/04/2008 – 07/08/2012)
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(Data obtained from UK: 12/04/2008 – 07/08/2012)
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(Data obtained from UK: 12/04/2008 – 07/08/2012)
## Estimated NIG parameters, skews, kurtoses, Kolmogorov-Smirnov (K-S) and Chi-square ($\chi^2$) test results for NIG FX models (Data obtained from UK: 12/04/2008 – 07/08/2012)

<table>
<thead>
<tr>
<th>FX</th>
<th>Alpha</th>
<th>Beta</th>
<th>Delta</th>
<th>Mu</th>
<th>Skew</th>
<th>Kurtosis</th>
<th>KS p-value</th>
<th>KS D-value</th>
<th>CS P-value</th>
<th>CS stats</th>
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</thead>
<tbody>
<tr>
<td>AUD/JPY</td>
<td>0.33</td>
<td>-0.23</td>
<td>5.07</td>
<td>84.3</td>
<td>-1.53</td>
<td>2.49</td>
<td>0.4</td>
<td>0.04</td>
<td>1</td>
<td>158.1</td>
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<tr>
<td>CHF/JPY</td>
<td>0.54</td>
<td>0.26</td>
<td>7.31</td>
<td>82.41</td>
<td>0.76</td>
<td>1.16</td>
<td>0.22</td>
<td>0.047</td>
<td>1</td>
<td>169.28</td>
</tr>
<tr>
<td>EGP/EUR</td>
<td>18215.6</td>
<td>18011.2</td>
<td>0.004</td>
<td>0.1</td>
<td>0.38</td>
<td>-0.76</td>
<td>0.062</td>
<td>0.059</td>
<td>0.01</td>
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<tr>
<td>EUR/GBP</td>
<td>2194.25</td>
<td>-412.12</td>
<td>2.41</td>
<td>1.32</td>
<td>-0.01</td>
<td>-0.07</td>
<td>0.08</td>
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<tr>
<td>GBP/JPY</td>
<td>8.43</td>
<td>8.31</td>
<td>4.32</td>
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<td>-0.3</td>
<td>0.12</td>
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<td>1</td>
<td>100.5</td>
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<tr>
<td>NZD/USD</td>
<td>354.97</td>
<td>-342.22</td>
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<td>-0.98</td>
<td>0.44</td>
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<td>103.31</td>
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<tr>
<td>QAR/CHF</td>
<td>2152.2</td>
<td>-2092.4</td>
<td>0.02</td>
<td>0.37</td>
<td>-0.77</td>
<td>0.29</td>
<td>0.12</td>
<td>0.053</td>
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<td>304.05</td>
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<tr>
<td>QAR/EUR</td>
<td>1364.5</td>
<td>1022.2</td>
<td>0.07</td>
<td>0.12</td>
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<td>0.37</td>
<td>0.041</td>
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<td>359.56</td>
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<tr>
<td>SAR/CHF</td>
<td>2656.5</td>
<td>-2594.4</td>
<td>0.02</td>
<td>0.36</td>
<td>-0.77</td>
<td>0.28</td>
<td>0.16</td>
<td>0.05</td>
<td>1</td>
<td>329.36</td>
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<tr>
<td>SAR/EUR</td>
<td>2702.86</td>
<td>2331.53</td>
<td>0.054</td>
<td>0.099</td>
<td>0.19</td>
<td>-0.6</td>
<td>0.16</td>
<td>0.05</td>
<td>0.92</td>
<td>359</td>
</tr>
<tr>
<td>TND/CHF</td>
<td>1088.3</td>
<td>-1065.6</td>
<td>0.047</td>
<td>0.99</td>
<td>-0.76</td>
<td>0.28</td>
<td>0.341</td>
<td>0.042</td>
<td>1</td>
<td>169.02</td>
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<tr>
<td>TND/EUR</td>
<td>1014.79</td>
<td>878.64</td>
<td>0.153</td>
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<td>0.341</td>
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<td>254.9</td>
</tr>
</tbody>
</table>
Observations...

Daily closing prices (12/04/2008 – 07/08/2012) of AUD/JPY, CHF/JPY, GBP/JPY, NZD/USD, QAR/CHF, SAR/CHF, SAR/EUR, TND/CHF, TND/EUR are excellent fits while EGP/EUR and EUR/GBP are good fits with a Kolmogorov-Smirnov test p-value of 0.062 and 0.08 respectively.

Impossible to estimate NIG parameters (by maximum likelihood) for JPY/CHF but CHF/JPY was an excellent fit.

Thus, while the stochastic properties of an exchange rate can be completely modeled with a probability distribution in one direction, it may be impossible the other way around.
Major conclusions

• The distribution of closing prices of stocks, world indexes and exchange rates can be modeled with the normal inverse Gaussian distribution; despite different time horizons

• Modeling the distribution of an exchange rate in one direction does not mean it can be modeled in the other direction

N/B: Chi-square test results depend on how intervals are chosen, number of variables etc and can basically be ignored in our study.
References

- Teneng, D.: Modeling foreign exchange closing prices with normal inverse Gaussian distribution. (Submitted)
Thank you for listening

ESF Grant No.8802 and Estonian Doctoral School in Mathematics and Statistics.